

THE
MÉTHOD of FLUXIONS

Applied to a select Number of
USEFUL PROBLEMS, &c.

THE
METHOD OF FUSIONS



USEFUL PROBLEMS, &c.

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THE
METHOD of FLUXIONS

Applied to a select Number of
USEFUL PROBLEMS:

TOGETHER WITH
The Demonstration of Mr. COTES's FORMS
of FLUENTS in the Second Part of his
LOGOMETRIA;

The ANALYSIS of the PROBLEMS in his
SCHOLIUM GENERALE;

AND
An EXPLANATION of the principal PROPOSITIONS of
Sir ISAAC NEWTON's PHILOSOPHY.

BY
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Late Professor of Mathematics in the University
of CAMBRIDGE.

Illustrated with TWELVE COPPER PLATES.

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M D C C L V I.

THE
METHOD OF FLEXIONS

USEFUL PROBLEMS



THE PROBLEM OF THE FLEXIONS

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THE ANALYSIS OF THE PROBLEMS

SCHOLIUM GENERALE

AND

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ADVERTISEMENT.

THE following Pieces of Dr. Saunderson, having been long handed about in Manuscript, and some Things, copied from them, having appeared in Print, under other Names; it was thought proper to collect them into one Volume: As well in Justice to their excellent Author, as to prevent the Errors that are unavoidable, and ever growing, when Works of this Kind are often transcribed.

The Reader will every where discover in them the Hand of a great Master; that Distinctness and Perspicuity for which Dr. Saunderson was so justly celebrated, a judicious choice of Examples, a simple Analysis, and an elegant Construction. And if their Defects are supplied *vivâ voce* by a skilful Tutor, as they ought to be, this Treatise of Fluxions, if not the completest, may nevertheless be reckoned the best for Students in the Universities, of any yet published.

What the Doctor has given us upon Mr. Cotes's Logometria is particularly valuable; as, by his intimate Acquaintance with that extraordinary Person, he may be presumed to have understood his Writings better than any one at
that

ADVERTISEMENT.

that Time living; Dr. Smith only excepted, to whose superior Genius and faithful Care the World is so much indebted for the Improvement, as well as the Preservation, of Mr. Cotes's Works.

*This Collection furnishes likewise an Instance remarkably curious, and such as may not again happen; how far the Faculties of the Mind, Imagination, and Memory, properly cultivated, may supply the Want of a Sense * which, but for this Instance, might seem absolutely necessary to the Acquisition of Mathematical Learning.*

* See Dr. Saunderson's Life prefixed to his *Algebra*.

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AN
INTRODUCTION.

- I. *Of the COMPOSITION and RESOLUTION of FORCES.*
II. *Of the DESCENT of HEAVY BODIES.*
III. *Of POWERS and their INDEXES.*

Plate I. Fig. Q.

N. B. **B**Y the Determination of a Force in the following Theorem is either meant its Direction, or any other Line parallel to it. Thus if a Force acts from A towards B, and any Line as EF be drawn parallel to AB, and F be situated the same Way with respect to E that B is with respect to A; then may the Line EF be said to signify the Determination of the Force acting from A, and the Line FE a contrary Determination.

PROP. I.

If there be a Triangle EFG whereof two Sides EF and FG represent both the Quantities and Determinations of two Forces acting from a given Point as A; I say then that the third Side EG will represent both the Quantity and Determination

x *The COMPOSITION and*

tion of a third Force, which acting from the same Point A, will be equivalent to the other two.

For drawing AB equal and parallel to EF, and AC equal and parallel to FG, and completing the Parallelogram ABDC, the Lines AB and AC will represent both the Quantities and Directions of the two Forces acting from A, which two Forces will both together be equivalent to the single Force AD, as is evident from the first *Corollary* of *Newton's Laws of Motion*. Now since EF and FG are equal and parallel to AB and BD respectively, the Angle EFG will be equal to the Angle ABD; therefore by the fourth *Prop.* of the first of the *Elements*, the Side EG will be equal to the Side AD, and the Angle FEG to the Angle BAD; therefore since the Lines EF and FG have the same Determination with the Lines AB and BD, the Line EG will be also parallel to AD; therefore the Line EG will represent both the Quantity and Determination of the Force AD. Q. E. D.

COROL. I.

In like manner if there be ever so many Forces acting from the Point A, and if the Quantities and Determinations of these Forces be represented by so many contiguous Lines EF, FG, GH, HI, the Line EI, drawn from the first Term E to the last I, will represent both the Quantity and Determination of a simple Force, which acting from A will be equivalent to all the former put together.

RESOLUTION of FORCES. xi

For the Forces EF and FG are equivalent to the Force EG by the Proposition; and for the same Reason the Forces EG and GH are equivalent to the Force EH ; and the Forces EH and HI to the Force EI ; therefore the Forces EF, FG, GH, HI are equivalent to the Force EI .

COROL. II.

If the last Point coincides with the first Point E , it is an Argument that the Forces at A will have no Effect, but will only sustain one another in *Aequilibrio*.

COROL. III.

On the other hand, if $EFGHI$ be a Polygon, whereof any one Side as EI represents both the Quantity and Determination of a simple Force acting from a given Point as A , the other Sides EF, FG, GH, HI will represent the Quantities and Determinations of so many other Forces, which acting from the same Point A will be all together equivalent to the former simple one.

Of the DESCENT of HEAVY BODIES.

PROP. II.

The Velocity generated in any Quantity of Matter q by the constant Action of an uniform Force as r continued for the Time t , will be as the Quantity $\frac{rt}{q}$. For it will be, *caeteris paribus*, as r directly, and as t directly, and as q inversely. Q. E. D.

COROL. I.

If the moving Force r be as the Quantity of Matter q , the Quantity $\frac{r}{q}$ will be constant, in which Case the Velocity generated will be as the Time t , and in the same Time will always be the same.

COROL. II.

Therefore if any Number of Bodies, how different soever in their Kinds and Quantities of Matter, be acted upon by Forces proportionable to their Matter, they will be equally accelerated, and consequently will pass through equal Spaces in equal Times.

COROL. III.

And *vice versa*, if any Number of Bodies, how heterogeneous soever, be acted upon by different Forces, and be found to be equally accelerated, that is, to pass through equal Spaces in equal Times.

Times ; it is an Argument that the Forces whereby these Bodies are moved are proportionable to the Quantities of Matter they contain.

SCHOLIUM.

The Quantity r is usually called the *Vis motrix*, and the Quantity $\frac{r}{q}$ the *Vis acceleratrix* ; because the former is always proportionable to the Motion generated in any given Time, and the latter to the Velocity ; and as the specific Gravity of a Body is not its Gravity absolutely, but its Gravity with respect to its Magnitude, so the *Vis acceleratrix* of a Body is not the *Vis motrix* absolutely, but the *Vis motrix* with respect to its Matter ; and in this Sense of the Word our Proposition will stand thus, viz. The Velocity generated by the constant Action of an uniform Force continued for any Time, will be as the Product of the Time and the accelerating Force multiplied together : And therefore in Cases where the accelerating Force is always the same, the Velocity generated will be as the Time.

PROP. III.

That the Weights of all Bodies are proportionable to their Quantities of Matter.

This appears from the equal Accelerations of Bodies of all Kinds in Vacuo, and is further confirmed by Experiments made upon the Oscillations of Pendulums, whereof more hereafter.

PROP. IV.

That the Gravity betwixt the Earth and its Parts is mutual and equal.

To demonstrate this, let the Globe of the Earth be distinguished by the Mind into two Segments equal or unequal. Now if either of these Segments should gravitate more upon the other, than the other upon it, there would be a greater Pressure one Way than the other, and the Earth giving Way to the greater Pressure, would be forced out of its Place *ad infinitum*; not to say that upon this Supposition a Body at Rest would be able to put itself into Motion without the Action of any external Force whatever. Q. E. A. See Cotes's *Preface to the Principia*, and Newton's general Scholium to the *Laws of Nature*.

PROP. V.

That the Forces wherewith Bodies attract the Earth are proportionable to their Quantities of Matter.

For the Forces wherewith Bodies attract the Earth are equal to the Forces wherewith the Earth attracts them by the fourth *Prop.* But these latter Forces are proportionable to the Quantities of Matter in the Bodies attracted by *Prop.* 3, therefore the former are so too. Q. E. D.

COROL. I.

As the Force wherewith every Body either attracts the Earth, or is attracted by it, is owing,
caeteris

cæteris paribus, to its Matter as such, and not to any particular Constitution of the Body, so neither is the Force wherewith the Earth attracts Bodies, or is attracted by them, owing to any particular Constitution of the Earth, or to any magnetic Body in its Center, but to the Mass of Matter whereof it consists.

COROL. II.

Since the Attraction of the whole Earth arises from the Attraction of all its Parts; and since all Bodies, as well those that lie loose upon its Surface as those that adhere to it, may be looked upon as Parts of the whole Earth, it follows that all the Bodies whereof our Earth is composed, mutually and equally attract one another, though this Attraction be too small to have any sensible Effect.

PROP. VI.

If a heavy Body as q pressed by its own Weight r descends through the Space s in the Time t , and at last acquires the Velocity v ; I say that the Space s will be but Half the Descent it would have made, had it moved all the Time t with the same Velocity v .

For since the Velocity of a falling Body increases uniformly with the Time of the Fall, and since between the first Velocity which was 0, and the last which was v , an arithmetic Mean is $\frac{1}{2}v$, it follows that a Body moving uniformly with Half the Velocity v will pass through the Space s in

the Time t ; therefore a Body moving uniformly with the whole Velocity will pass through twice that Space in the same Time. Q. E. D.

COROL. I.

Since a Body moving uniformly with the Velocity v , describes the Space $2s$ in the Time t , it follows, that $2s$, and consequently s will be as vt ; that is, the Space through which a heavy Body descends will be as the Time of the Descent, and the last Velocity multiplied together.

COROL. II.

Since by the Scholium to the second *Prop.* v is as rt , we shall have vt and consequently s as rtt .

COROL. III.

For the same Reason, viz. that v is as rt , we shall have vv as rtt , and consequently as rs ; whence s will be as $\frac{vv}{r}$.

COROL. IV.

From the three foregoing *Corollaries*, and also from hence that v is as rt , may be deduced the following Canons.

$$1. r \text{ is as } \frac{s}{tt}, \text{ or as } \frac{v}{t}, \text{ or as } \frac{vv}{s}.$$

$$2. s \text{ is as } rtt \text{ or as } vt, \text{ or as } \frac{vv}{r}.$$

$$3. t$$

3. t is as $\frac{s}{v}$, or as $\frac{\sqrt{s}}{\sqrt{r}}$, or as $\frac{v}{r}$.

4. v is as rt , or as $\sqrt{rs}$, or as $\frac{s}{t}$.

COROL. V.

If the *Vis acceleratrix* r be given, it may be struck out of the Canons, and then they will stand thus :

2. s is as tt , or as tv , or as vv .

3. t will be as $\frac{s}{v}$, or as $\sqrt{s}$, or as v .

4. v is as t , or as $\sqrt{s}$, or as $\frac{s}{t}$.

SCHOLIUM.

At the Surface of our Earth, a heavy Body descends through $15\frac{1}{12}$ French Feet or $16\frac{1}{2}$ * English Feet in a Second of Time ; therefore by this *Prop.* in a Second of Time a heavy Body acquires a Velocity which would carry it uniformly through $32\frac{2}{5}$ English Feet in the same Time. Let then this Quantity represent this Velocity, and every other Velocity will be represented by the Space through which it would carry a Body uniformly in a Second of Time. By the Help of these Data, and the Canons delivered in the fifth *Corollary*, if in the Case of any other particular Descent, at or near the Surface of the Earth, either the Time or Space, or the Velocity be given,

* $16\frac{7}{10}$ quam proxime.

the

the rest may easily be found. As for Instance, let it be required to determine how long, and how far a heavy Body must fall to acquire the Velocity of a Sound, that is, to acquire a Velocity which will carry a Body uniformly through 1142 *English* Feet in a Second of Time.

First, To determine the Time, I find in the fifth *Corollary*, t is as v ; therefore I say, as $32 \frac{2}{5}$ is to 1142, so is one Second of Time wherein the former Velocity was acquired to the Time sought wherein the latter Velocity will be acquired.

Secondly, As to the Space, I find s to be as $v v$; and therefore say, as $32 \frac{2}{5}^2$ is to 1142^2 so is $16 \frac{1}{5}$ Feet through which a heavy Body must fall to acquire the former Velocity, to the Space sought, through which it must fall to acquire the latter Velocity.

If in any particular Case the *Vis acceleratrix* that causes the Descent be different from that of common Gravity, then of the four Quantities r, s, t, v , two must be given to find the rest, and the Canons must be taken out of the fourth *Corollary*.

PROP. VII. Plate I. Fig. R.

The Force whereby a Body endeavours to descend freely upon an inclined Plane is as the Elevation of the Plane directly, and as its Length inversely.

For in the right-angled Triangle ABC, let the Hypothenuse AC represent an inclined Plane whose highest Point is A, and lowest C, and let
BC

BC be parallel, and AB perpendicular to the Horizon; then will AB be what is called the Elevation of the Plane AC. Let the same Line AB also represent the absolute Weight of a Body lying upon the Plane AC, whereby, if permitted, it would descend perpendicularly: Then drawing BD perpendicular to AC, the Force AB will be equivalent to two Forces AD and DB, whereof AD will be wholly spent in accelerating the Motion of the Body along the Plane AC, but the other BD will have no Effect, as being opposed by the perpendicular Reaction of the Plane; therefore the absolute Weight of the Body will be to its Endeavour to descend along the Plane AC, as AB is to AD, or by similar Triangles as AC to AB, or as 1 to $\frac{AB}{AC}$: Therefore if the absolute Weight of any Body be called 1, its Endeavour to descend obliquely along the Plane AC will be represented by the Quantity $\frac{AB}{AC}$. Q.E.D.

COROL. I.

The accelerating Force in every Part of the same Plane AC will be the same; for it will always be represented by the same Quantity $\frac{AB}{AC}$; and therefore whatever has been demonstrated in the sixth *Prop.* and its *Corollaries* concerning the perpendicular Descent of heavy Bodies, will be equally applicable to the Case of oblique Descent, making $r = \frac{AB}{AC}$, and $s = AC$.

COROL.

COROL. II.

The Velocity acquired in falling from A to C will be in a subduplicate Ratio of A B the Elevation of the Plane. For by the fourth *Corollary* of the sixth *Prop.* v is as $\sqrt{rs}$; but in our Case $r = \frac{AB}{AC}$, and $s = AC$; therefore $rs = AB$, and $\sqrt{rs} = \sqrt{AB}$; therefore v is as $\sqrt{AB}$.

COROL. III.

Therefore so long as the Elevation of a Plane is the same, the Velocity acquired in falling through the Length of the Plane will always be the same, be that Length what it will.

COROL. IV.

Therefore in falling from the same Place A to the same horizontal Line BC, the same Velocity will always be acquired, whether the Descent be made obliquely upon an inclined Plane AC, or perpendicularly in the Line AB.

COROL. V.

The Time of oblique Descent from A to C will be as the Length of the Plane AC directly, and in a subduplicate Ratio of the Elevation AB inversely. For by the fourth *Corollary* of the sixth *Prop.* t is as $\frac{s}{v}$, that is, in our Case as $\frac{AC}{\sqrt{AB}}$.

COROL.

COROL. VI.

Therefore if there be ever so many Planes having all the same Elevation, the Times of oblique Descent through those Planes will be as their Lengths.

COROL. VII.

If s be the Space through which a heavy Body falls in a Second of Time, and if m be a mean Proportional between the Space s , and the Elevation AB of any Plane; then will the Velocity acquired in falling obliquely from A to C be such as will carry a Body uniformly through the Space $2m$ in a Second of Time. For by the fifth *Corollary* of the sixth *Prop.* the Velocity acquired in falling through s will be to the Velocity acquired in falling through AB , and consequently through AC , as $\sqrt{s}$ is to $\sqrt{AB}$, or as s to m : But the former Velocity will describe the Space $2s$ in a Second of Time; therefore the latter will describe the Space $2m$ in the same Time.

COROL. VIII.

Supposing all Things as in the foregoing *Corol.* I say that as m is to AC , so is 1 Second of Time to the Time of oblique Descent from A to C . For considering the Line s as a Plane whose Elevation is equal to its Length, it follows from the fifth *Corollary* that the Time of perpendicular Descent through s , that is, 1 Second of Time will be to the

xxii Of POWERS and their INDEXES.

the Time of oblique Descent through A C as $\frac{s}{\sqrt{s}}$
 is to $\frac{AC}{\sqrt{AB}}$, or as $\sqrt{s}$ to $\frac{AC}{\sqrt{AB}}$, or as $\sqrt{s} \times \sqrt{AB}$
 to A C, or as m to A C.

Of POWERS and their INDEXES.

x^6 signifies $xxxxxx$

x^5 - - - $xxxxx$

x^4 - - - $xxxx$

x^3 - - - xxx

x^2 - - - xx

x^1 - - - x

x^0 - - - 1

x^{-1} - - - $\frac{1}{x}$

x^{-2} - - - $\frac{1}{xx}$

x^{-3} - - - $\frac{1}{xxx}$

x^{-4} - - - $\frac{1}{xxxx}$

x^{-5} - - - $\frac{1}{xxxxx}$

x^{-6} - - - $\frac{1}{xxxxxx}$

OBSER-

Of POWERS and their INDEXES. xxiii

OBSERVATIONS.

1. Every succeeding Power is generated by dividing the Power immediately preceding by the Root x , and every succeeding Index is generated by subtracting Unity from the foregoing.

2. Whatever Number is the Index of any Power, its Negative will be the Index of the reciprocal of that Power, or of Unity divided by that Power.

3. The Addition of Indexes answers to the Multiplication of Powers; that is, if the Multiplier, and Multiplicand be any Powers of the same Quantity, the Index of the Multiplier added to the Index of the Multiplicand will give the Index of the Product.

4. The Subtraction of Indexes answers to the Division of their Powers; that is, If the Index of the Divisor be subtracted from the Index of the Dividend, the Remainder will be the Index of the Quotient.

5. The Duplication, Triplication, &c. of Indexes answers to the Squaring, Cubing, &c. of their Powers.

6. The bisecting, trisecting, &c. of Indexes answers to the extracting the Square or Cube Root of their Powers.

7. Any Root of any Power may be expressed by a fractional Index whose Numerator signifies the Power, and its Denominator the Root.

8. Surds

xxiv *Of Powers and their INDEXES.*

8. Surds may often be reduced to more simple Terms by a Reduction of their Indexes.

9. Surds may be reduced to the same Denomination by a Reduction of their Indexes.

10. The third and fourth Observations will be true in fractional as in integral Powers.

11. In the Common Tables of Logarithms, Unity being assumed for the Logarithm of 10, if 10^m is equal to any Number N, then m is the Logarithm of N— and the Reason of the Operations by Logarithms is manifest.

OF

OF THE
ALGORITHM of FLUXIONS.

DEFINITION.

LET AB represent any Moment of Time, whether finite or infinitely small it matters not, terminated by the two Instants A and B : Let x be the Value of any flowing or growing Quantity at the Instant A , whose Velocity at that Instant is such, that if it was to flow during the whole Moment AB with this Velocity, it would gain a certain Increment represented by $\dot{x}$; then is this Quantity $\dot{x}$ called the *Fluxion* of x at the Instant A , when the Value of the flowing Quantity was x .

SCHOLIUM.

From this Definition it appears, that if x flows uniformly, its Increment gained in the Time AB will be the same as its Fluxion above defined: But if x does not flow uniformly, *i. e.* if its Velocity at the Instant B be not the same as its Velocity at the Instant A , then its Increment gained in the Time AB will not be the same with its Fluxion above defined, but will differ more or less from

B

it,

it, according as the Time A B is greater or less: But if the Time A B be infinitely small, then though the Velocity of x at the Instant B be not the same, mathematically speaking, with the Velocity at the Instant A; yet the Difference being infinitely small in Respect of the whole Velocity, it may safely be neglected, where the finite Ratios of Fluxions are only considered; and so this Increment and the Fluxion above defined may be taken for one another, *i. e.* the Quantity x , for so small a Time, may be looked upon as flowing uniformly.

COROLL. I.

The Fluxion of a constant Quantity is nothing.

COROLL. II.

The Fluxion of the whole equals the Fluxion of all its Parts. Thus if r be a constant Quantity, and x, y, z flowing ones, the Fluxion of $r + x + y - z$ will be $\dot{x} + \dot{y} - \dot{z}$.

N. B. When different Fluxions are compared, they are all supposed to be generated uniformly in the same Time.

Note also, That a constant Equality of Fluents always implies an Equality of Fluxions, but not *vice versa*, unless the Fluents be generated in equal Times.

LEMMA.

LEMMA.

If two Quantities v and x , arithmetically expressed, be supposed to flow uniformly, the Product of their Multiplication will not flow uniformly, but with a Velocity equally accelerated; that is, its continual Increments gained in successive equal Times will not be equal, but in arithmetical Progression.

DEMONSTRATION.

Let the Time AD be divided into the equal Parts AB , BC , CD , and let v and x be the Values of the flowing Quantities at the Instant A ; then because these Quantities are supposed to flow uniformly, their Values will be at the Instant B , $v + \dot{v}$ and $x + \dot{x}$; at the Instant C , $v + 2\dot{v}$, and $x + 2\dot{x}$; at the Instant D , $v + 3\dot{v}$, and $x + 3\dot{x}$: Therefore their Product will be, at the Instant A , vx ; at the Instant B , $vx + v\dot{x} + x\dot{v} + \dot{v}\dot{x}$; at the Instant C , $vx + 2v\dot{x} + 2x\dot{v} + 4\dot{v}\dot{x}$; at the Instant D , $vx + 3v\dot{x} + 3x\dot{v} + 9\dot{v}\dot{x}$. Subtract now the Product at the Instant A from the Product at the Instant B , and there will remain $v\dot{x} + x\dot{v} + \dot{v}\dot{x}$ for the Increment gained by the Product in the Time AB . Subtract again the Product at the Instant B , from the Product at the Instant C , and there will remain $v\dot{x} + x\dot{v} + 3\dot{v}\dot{x}$ for the Increment gained in the Time BC : Lastly, subtract the Product at the Instant C from the Product at the Instant D , and there will remain $v\dot{x} + x\dot{v} + 5\dot{v}\dot{x}$ for the Increment gained in the

B 2

Time

Time C D: Therefore the continual Increments, gained by the Products in the successive equal Times A B, B C, C D, will not be equal, but in arithmetical Progression.

N. B. If the Times A B, B C, C D, be infinitely small, the Quantities $\dot{v}\dot{x}$, $3\dot{v}\dot{x}$, $5\dot{v}\dot{x}$ will be infinitely less than the other Parts of the Increments; as will easily appear by comparing them.

PROBLEM I.

To find the Fluxion of the Product arising from the continual Multiplication of any Number of flowing Quantities whatsoever.

CASE I.

Let v and x be the Values of two flowing Quantities at any given Instant of Time, and let these Quantities be supposed to flow uniformly as in the last Lemma; then it is plain that a Moment before the given Instant, their Values were $v - \dot{v}$, and $x - \dot{x}$, and their Product $v x - v \dot{x} - x \dot{v} + \dot{v}\dot{x}$; therefore the Increment gained by the Product in that Moment was $v \dot{x} + x \dot{v} - \dot{v}\dot{x}$: But the Increment gained by the Product in an equal Moment immediately following the given Instant was found to be $v \dot{x} + x \dot{v} + \dot{v}\dot{x}$; and this latter Increment, by the above-quoted Lemma, exceeds as much the true Fluxion of the Product as the other wants of it; therefore the true Fluxion of the Product $v x$ at the Instant of Time that the Factors are v and x is $v \dot{x} + x \dot{v}$. Q. E. I.

Other

Otherwise thus.

The Increment gained by the Product in the Time A B (see the *Lemma*) is $v\dot{x} + x\dot{v} + \dot{v}\dot{x}$. Let this Time A B be infinitely small; then it is evident that the Quantity $\dot{v}\dot{x}$, being infinitely less than the rest, may be neglected for its Insignificance; so that both the Increment and the Fluxion of the Product will be $v\dot{x} + x\dot{v}$. Increase now again the Time A B from an infinitely small to a finite Moment, and $v\dot{x}$ and $x\dot{v}$ will increase with the Time, as is evident from the Definition already given of a Fluxion: Therefore $v\dot{x} + x\dot{v}$ will still be the Fluxion of the Product vx .

CASE II.

Let there now be three flowing Quantities, v , x and y , and the Fluxion of the Product $vx y$, or of the Product $vx \times y$ will be the Fluxion of vx multiplied into y together with the Fluxion of y multiplied into vx , by the last Case: But the Fluxion of vx was there found to be $v\dot{x} + x\dot{v}$, and this multiplied into y gives $vy\dot{x} + xy\dot{v}$; and the Fluxion of y being multiplied into vx gives $vx\dot{y}$; therefore the whole Fluxion of the Product $vx y$ is $vx\dot{y} + vy\dot{x} + xy\dot{v}$.

CASE III.

In like Manner, if there be four flowing Quantities, the whole Fluxion of the Product $vx y z$, or the Product $vx y \times z$ will be found to be $vx y \dot{z} + vx \dot{y} z + vy \dot{x} z + xyz \dot{v}$. Q. E. I.

B 3

COROLL.

COROLL.

If a be a standing Quantity, and v a variable one, the Fluxion of the Product av will be $a\dot{v}$.

PROBLEM II.

To find the Fluxion of any Power of any flowing Quantity, whether the Index of that Power be integral or fractional, affirmative or negative.

CASE I.

Let v , x , y and z , in the last Problem, be supposed all equal to one another; then will $vx = x^2$, and its Fluxion $v\dot{x} + x\dot{v}$ will be $2\dot{x} \times x$; therefore the Fluxion of x^2 will be $2\dot{x} \times x$, or $2\dot{x} \times x^{2-1}$. Again, $vxy = x^3$, and its Fluxion $vxy\dot{v} + v\dot{y}x + v\dot{x}y$ will be $3\dot{x} \times x^2$; therefore the Fluxion of x^3 will be $3\dot{x} \times x^2$, or $3\dot{x} \times x^{3-1}$. Again $vxyz = x^4$; therefore the Fluxion of x^4 will be $4\dot{x} \times x^3$. And universally, the Fluxion of x^m will be $m\dot{x} \times x^{m-1}$, provided the Index m be integral and affirmative.

CASE II.

Let it now be required to find the Fluxion of x^{-m} or $\frac{1}{x^m}$: In order to which, make $\frac{1}{x^m} = z$; then will $z \times x^m = 1$; and taking the Fluxions on both Sides, we have $\dot{z} \times x^m + m z \dot{x} \times x^{m-1} = 0$, (for the Fluxion of 1 is 0); therefore $\dot{z} \times x^m = -m z \dot{x} \times x^{m-1} = \frac{-m \dot{x} \times x^{m-1}}{x^m} = -m \dot{x} \times x^{-1}$; there-

therefore $\dot{z} = -m\dot{x} \times \frac{x^{-1}}{x^m} = -m\dot{x} \times x^{-m-1}$. So that from these two last Cases it is easy to perceive, that if the Index m be a whole Number, whether affirmative or negative, the Fluxion of x^m will be $m\dot{x} \times x^{m-1}$.

CASE III.

Let it now be required to find the Fluxion of $x^{\frac{m}{n}}$.

To do this, let $x^{\frac{m}{n}} = z$; then will $z^n = x^m$, from the fifth Observation of *Powers* and their *Indexes*; therefore $n\dot{z} \times z^{n-1} = m\dot{x} \times x^{m-1}$, from the two foregoing Cases; therefore (dividing Equals by Equals) we have $n\dot{z} \times z^{-1} = m\dot{x} \times x^{-1}$, and $\dot{z} \times z^{-1} = \frac{m}{n}\dot{x} \times x^{-1}$. Multiply both Sides by z , and we have $\dot{z}$ (or the Fluxion sought) $= \frac{m}{n}\dot{x} \times z \times x^{-1} = \frac{m}{n}\dot{x} \times x^{\frac{m}{n}-1}$. So that at last it appears that whether the Index be affirmative or negative, integral or fractional, the Fluxion of x^m will be $m\dot{x} \times x^{m-1}$.

COROLL. I.

The Fluxion of this Quantity $\frac{x^m \times y^n}{z^p}$, or $x^m \times y^n \times z^{-p}$, by the two foregoing Problems, will be $m\dot{x} \times x^{m-1} \times y^n \times z^{-p} + n\dot{y} \times y^{n-1} \times x^m \times z^{-p} - p\dot{z} \times z^{-p-1} \times x^m \times y^n$.

COROLL. II.

The Fluxion of any Fraction $\frac{x}{y}$ will be $\frac{\dot{x} \times y - \dot{y} \times x}{yy}$; as will appear, either from the foregoing *Corollary*, by considering $\frac{x}{y}$ as $x^1 \times y^{-1}$; or rather thus: Make $\frac{x}{y} = z$; then will $yz = x$, and $y\dot{z} + z\dot{y} = \dot{x}$; therefore $y\dot{z} = \dot{x} - z\dot{y} = \dot{x} - \frac{x\dot{y}}{y} = \frac{y\dot{x} - x\dot{y}}{y}$; therefore $\dot{z}$ (or the Fluxion sought) $= \frac{y\dot{x} - x\dot{y}}{yy}$.

Of the Method of drawing TANGENTS.

PROB. III. Fig. I.

It is required to draw a Tangent to any given Point M as of any given Curve, as A M, whose Axis is A P.

SOLUTION.

Draw two parallel Ordinates M P, mp ; draw also the Chord M m , and produce it till it meets the Axis (produced also if need be) in T, and complete the Parallelogram P p m R; then will the Triangle R M m be similar to the Triangle P M T, and we shall have R M to R m (or P p) as M P to P T; whence $P T = M P \times \frac{P p}{R M}$. Let

now

FLUXIONS exemplified.

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now the Ordinate mp be conceived to move always parallel to itself, till at last it arrives at MP , or at least to come infinitely near it; and during this whole Motion, let the Line MT be supposed to turn so upon the Point M , as always to pass through m ; then it is plain that the Line MT will at last become a Tangent to the Curve in the Point M , and the Line PT a *Subtangent*; and if we call AP, x , MP, y ; we shall have at last

$$Pp = \dot{x}, RM = \dot{y}, \text{ and the Subtangent } TP = \frac{y\dot{x}}{\dot{y}}.$$

Compute therefore, from the Nature of the Curve, the Quantity $\frac{y\dot{x}}{\dot{y}}$, and you will have the Subtangent PT : As in the following Examples.

EXAMPLE I.

Supposing all Things as before, let AM be the common Parabola, whose Parameter let be p ; then we shall have $p x = y^2$, and $p \dot{x} = 2 y \dot{y}$, and $\dot{x} = \frac{2 y \dot{y}}{p}$, and $y \dot{x} = \frac{2 y^2 \dot{y}}{p}$, and $\frac{y \dot{x}}{\dot{y}}$, or the Subtangent $PT, = \frac{2 y y}{p} = \frac{2 p x}{p} = 2 x = 2 AP$; as we find in the *Conic Sections*.

EXAMPLE II.

Let the Nature of the Curve be expressed by this Equation $p x^m = y^n$; Then we shall have $p m \dot{x} x^{m-1} = n \dot{y} \times y^{n-1}$, and $m \dot{x} \times x^{m-1} = n \dot{y} \times y^{n-1}$,
or

or $\frac{m\dot{x}}{x} = \frac{n\dot{y}}{y}$: Hence $m\dot{x} = \frac{nxy}{y}$, and $my\dot{x} = nxy\dot{y}$,

and $\frac{my\dot{x}}{y} = nx$, and $\frac{y\dot{x}}{y}$, or PT , $= \frac{nx}{m} = \frac{n}{m} \times AP$.

If $px = y$; then will $PT = AP$, and AM will be a right Line, because $x : y :: 1 : p$.

Call the *Power* of the *Hyperbola* p , and (by *L'Hospital's* Conic Sect. Art. 101.) $\frac{p}{x} = y$; but $\frac{p}{x} = p \times \frac{1}{x} = p \times x^{-1}$; therefore $p \times x^{-1} = y^1$.

Then will PT (*Fig. 3.*) $= -AP$, the Sign — signifying that PT the Subtangent does not lie on the same Side the Ordinate PM with the Abscisse AP . This is agreeable to *Conic Sections* Art. 107, where C answers to our A , H to our P , D to our T .

If $\frac{p}{x^3} = y^4$; then $p \times x^{-3} = y^4$, and $PT = -\frac{4}{3}AP$.

EXAMPLE III. Fig. 2.

Let AMB be an Ellipse: Call AC, t ; CB, c ; CP, x ; MP, y : Then we shall have, from the Nature of the Ellipse, $cc - \frac{c^2xx}{tt} = yy$, and consequently $-\frac{2c^2x\dot{x}}{tt} = 2y\dot{y}$; therefore $\dot{x} = -\frac{tt}{c^2x} \times y\dot{y}$, and $y\dot{x} = -\frac{tt}{c^2x} \times y^2\dot{y}$, and $\frac{y\dot{x}}{y}$, or PT , $= -\frac{tt}{c^2x} \times y^2$; the Sign — signifying only that the Subtangent PT is not now to be on the

FLUXIONS exemplified. 11

the same Side of the Ordinate with P C or x , but on the contrary Side. Further, since $y^2 = c^2 - \frac{c^2 x^2}{t^2} = \frac{c^2 t^2 - c^2 x^2}{t^2} = \frac{c^2}{t^2} \times \overline{t^2 - x^2}$, if instead of y^2 in the former Value of P T, we substitute its Equal $\frac{c^2}{t^2} \times \overline{t^2 - x^2}$, and we shall now have $P T = \frac{t t}{c^2 x} \times \frac{c^2}{t^2} \times \overline{t^2 - x^2} = \frac{t t - x x}{x}$; and if to this Value of P T we add C P or x , we shall have $C T = \frac{t t}{x}$; whence we shall have the following Proportion for finding the Point T, viz. C P : C A :: C A : C T.

Of the Doctrine de MAXIMIS et MINIMIS:

PROB. IV.

To determine when a flowing Quantity becomes a Maximum or a Minimum.

SOLUTION.

As the Fluxion of every flowing Quantity is affirmative or negative, according as that Quantity is upon the Increase or Decrease, it follows, that when a Quantity becomes a Maximum or a Minimum, its Fluxion will be equal to nothing. Take therefore the Fluxion of such a Quantity in general, and then suppose it equal to nothing, and the Equation will determine the Case wherein it is a Maximum or a Minimum.

EXAMPLE

EXAMPLE I.

Let it be required to divide a given Number, as a , into two such Parts that the Product of the Multiplication of those Parts may be the greatest possible.

Here putting x and $a - x$ for the two Parts, their Product will be $ax - xx$, whose Fluxion $a\dot{x} - 2x\dot{x}$ (I suppose) $= *$, and find $x = \frac{1}{2}a$; which shews that the Product of the two Parts will be greatest when they are equal to each other.

EXAMPLE II.

Let it be required to divide a given Number, as a , into two such Parts, that one Part being multiplied into the Square of the other may make the greatest Product possible.

Here I put x for one Part, and xx for its Square, and consequently $a - x$ for the other Part; then will the Product be $ax^2 - x^3$ whose Fluxion $2ax\dot{x} - 3x^2\dot{x}$ (I suppose) $= *$; and then find $x = *$ or $\frac{2}{3}a$, which shews that if the Part to be squared be $\frac{2}{3}$ of the whole, and consequently the other Part $\frac{1}{3}$, the Product will be the greatest possible: And since in the foregoing Equation, x was also found $= *$, that shews, that if x be less than $\frac{2}{3}a$, the Product will be least when x (or the Part to be squared) $= *$.

EXAMPLE III.

It is required to determine when the following Quantity becomes a Maximum or a Minimum, viz.
 $x^3 - 18x^2 + 96x$.

Here

Here we have $3x^2 \dot{x} - 36x \dot{x} + 96 \dot{x} = *$, and consequently $xx - 12x + 32 = *$, whence $x = 4$ or 8 ; therefore the Quantity proposed is a *Maximum* in one Case, and a *Minimum* in the other. But to determine distinctly to which Case the *Maximum* belongs, it must be observed, that the Fluxion of the Quantity proposed is either affirmative or negative, according as the Quantity $xx - 12x + 32$ is so: But this Quantity when $x = *$ is affirmative; therefore the Quantity proposed is upon the Increase till $x = 4$, when it becomes a *Maximum*; after this it decreases till $x = 8$, in which Case it becomes a *Minimum*, then it increases *ad infinitum*.

EXAMPLE IV.

To find the greatest Random of a Ball shot with a given Velocity. Fig. 4.

Let ABC be the Direction of the Ball, and let the Force given it at A be such as alone would have carried it from A to B, whilst the Ball by its Gravity alone would have descended from A through a known Space as s ; then will AB and s be constant Quantities, because the Velocity of the Ball at A, and the Force of Gravity are supposed the same, whatever be the Direction ABC. Let the horizontal Line AD be the Random of the Ball, and draw DC and BE Perpendiculars to AD. Lastly compleat the Parallelogram ACDF. This done, it is evident that whilst the Ball describes the curve Line AGD, it would, by its

its projectile Force alone have been carried from A to C, and by its Gravity alone from A to F : For the Force of Gravity acting in a Line parallel to CD will neither help nor hinder the Motion of the Ball towards that Line CD ; and the same may also be said with respect to the projectile Force, and the Line DF ; whence it follows, that whilst the Ball by its projectile Force alone would have passed through the Space AC, by its Gravity alone it would have descended through the Space AF or CD ; and since the Spaces through which heavy Bodies fall from Rest are as the Squares of the Times, it follows that as the Square of the Time wherein the Ball by its projectile Force alone would have described the Space AB, is to the Square of the Time wherein the same would have described the Space AC (*i. e.* as $\overline{AB}^2$ is to $\overline{AC}^2$ or as $\overline{AE}^2$ to $\overline{AD}^2$) so is s to CD ; therefore $CD = s \times \frac{\overline{AD}^2}{\overline{AE}^2}$.

But as AE is to AD so is BE to CD ; therefore CD is also equal to $BE \times \frac{AD}{AE}$; therefore $BE \times \frac{AD}{AE} = s \times \frac{\overline{AD}^2}{\overline{AE}^2}$; therefore $AD = \frac{AE \times EB}{s}$; therefore the Random AD will be the greatest when $\frac{AE \times EB}{s}$ becomes a *Maximum*, *i. e.* when AE $\times$ EB becomes a *Maximum*.

Call AB, r ; BE, x ; AE, y ; and in the present Case we shall have $x \times y + y \times x = *$.

But $xx + yy = rr$ a constant Quantity; therefore
 $2x\dot{x} + 2y\dot{y} = 0$; therefore $\dot{y} = -\frac{x\dot{x}}{y}$; there-
fore $\dot{y} \times x = -\frac{x^2\dot{x}}{y}$ therefore $\dot{x} \times y + \dot{y} \times x = \dot{x} \times y -$
 $\frac{x^2\dot{x}}{y} = \frac{yy\dot{x} - x^2\dot{x}}{y} = \frac{rr\dot{x} - 2x^2\dot{x}}{y} = 0$;
therefore $rr - 2x^2 = 0$; therefore $xx = \frac{1}{2}rr$,
and $yy = \frac{1}{2}rr$; therefore $x = y$: Therefore to
give the Ball the greatest Random possible, the
Piece must be elevated to an Angle whose Sine and
Co-sine are equal, *i. e.* to an Angle of forty-five
Degrees. Q. E. I.

SCHOLIUM.

If the Piece be elevated to an Angle of above
45 Degrees, the Random of the Ball will be the
same as if it was elevated to an Angle as much less
than 45 Degrees; because the Quantity $\frac{AE \times EB}{s}$
will be the same in both Cases.

EXAMPLE V. Fig. 5.

*To find the most commodious Situation of a Plane to be
turned about an Axis by a Wind whose Course is in
the Direction of that Axis.*

Let both the Plane of this Page, and the Axis
of Motion, supposed at some Distance under it, be
imagined in an horizontal Position. Let the Line
AB be parallel to the Axis, and let CBD be the
Plane in Question, in a Position perpendicular to
the

the Horizon, insisting with its lower Edge upon the Axis of its Motion, and represented by its Intersection with the Plane of the Paper in the Line CD . Let AE be perpendicular to CD , so that the Angle ABE may be the Inclination of the Plane to its Axis. Let EF be perpendicular, and CG be parallel to the Line AFB ; and let DG be perpendicular to GC . Let $\overline{AB^3}$ or the Cube of AB represent the absolute Force of the Wind upon the Plane CD , when directly exposed to its Current; then will the Force of the same Wind upon the oblique Plane CD be diminished in the Proportion of AB to AE or of $\overline{AB^3}$ to $AE \times \overline{AB^2}$; I say of the *same* Wind: But the Number of Particles now acting upon the oblique Plane CD will by its Obliquity be diminished in the Proportion of DC to DG , or of AB to AE , or of $AE \times \overline{AB^2}$ to $\overline{AE^2} \times AB$; wherefore upon the whole Matter, the Force of the Wind upon the oblique Plane CD must be represented by $\overline{AE^2} \times AB$. By this Force the Plane CD , if left to itself, would move not directly before the Wind, as when directly exposed to its Current, but in the Direction of the Line AE : This, I say, will be the Case if the Plane was absolutely at Liberty; but, in the present Case, the Plane is restrained from being moved in any other Direction but that of FE perpendicular to AB , by which Motion it carries round its Axis; and the *Conatus* of the Plane to move in the Direction AE will be to its *Conatus*

to

to move in the Direction FE, as AE to EF, or as AB to BE, or as $\overline{AE}^2 \times AB$ to $\overline{AE}^2 \times EB$: Therefore the *Conatus* of the Plane to move round its Axis must be represented by $\overline{AE}^2 \times EB$, and will be a *Maximum* when, supposing AB constant, the Quantity $\overline{AE}^2 \times EB$ is so.

Call AB, r ; BE, x ; and we shall have $\overline{AE}^2 \times EB = r^2x - x^3$; whose Fluxion $rr\dot{x} - 3xx\dot{x}$ (in this Case) $= \dot{x}$; whence $rr - 3xx = \frac{r}{x}$,

and $x = \frac{r}{\sqrt{3}}$: Therefore the Effect of the Wind

will be the greatest when the Plane is inclined to its Axis in an Angle whose Co-sine is to the Radius as 1 to $\sqrt{3}$; *i. e.* in an Angle of 54 Deg. 44 Min. This is the Case when the Plane is in a Situation perpendicular to the Horizon; and it will be the same in all others, provided that the Angle which the Plane makes with its Axis be the same. Q. E. I.

EXAMPLE VI. Fig. 6.

To construct the Frustum of a Cone, of a given Base and Altitude, which moving according to the Direction of its Axis, with its smaller End against the Parts of an uniform Fluid shall suffer the least Resistance possible from it.

Let the isosceles Triangle ABC represent a Cone generated by its Revolution about its Axis ADE. Let FDG be parallel to the Base; let FH and GH be Perpendiculars to the Sides AB and AC
C respec-

respectively ; and let FI be parallel to AE cutting the Base BC in I . This supposed, let us first inquire what Effect a Set of Particles moving in the Direction AE would have upon the Cone when at rest : In order to which, let the Point F be supposed to be struck by one of those Particles, and let AH represent the absolute Force with which the Particle moves, and with which it would strike the Base of the Cone if directly opposed to its Motion ; then it is plain that the Point F receiving the Stroke obliquely will only sustain the Force of a Stroke equal to FH . Let this Force FH be resolved into the two Forces FD and DH , and the former of these FD being sustained by another contrary Stroke GD coming from G , will have no Effect upon the Cone ; but the other Force DH meeting with no Opposition will have its full Effect upon the Cone to move it in the Direction DH , if not otherwise sustained ; therefore the Effect of a Particle at the Point F in the Conic Surface, will be to the Effect of the same Particle at the Point I in the Base, as DH is to AH , or as $AH \times HD$ to AH^2 , or (because of the similar Triangles HFD , HAF , where HD is to HF as HF to HA) as $\overline{HF}^2$ to $\overline{AH}^2$, or as $\overline{BE}^2$ to $\overline{AB}^2$. But what hath been said of the Points F and I will be equally true of any other two correspondent Points whatever ; therefore the Effect of any Number of Particles upon the Base of the Cone, will be to the Effect of the same Particles upon its Surface, as $\overline{AB}^2$ to $\overline{BE}^2$.

And

And since Action and Reaction are equal, if we now suppose the Parts of the Fluid at rest, and the Cone to move uniformly in it according to the Direction of its Axis, the Resistance the Base of the Cone would meet with, when moving in the Direction A E, will be to the Resistance its Surface would meet with, when moving with the same Velocity in the Direction E A, as $\overline{AB}^2$ to $\overline{BE}^2$. Let then $\overline{AB}^2$ represent the Resistance of the Base, and consequently $\overline{BE}^2$ that of the Surface, and let the Cone F A G be cut off; then will the Resistance of the Base B C in the Direction E A, be to the Resistance of the Base F G, as the Areae of those Bases, *i. e.* as the Squares of their Diameters, *i. e.* as $\overline{BC}^2$ to $\overline{FG}^2$, or as $\overline{AB}^2$ to $\overline{AF}^2$: Therefore since $\overline{AB}^2$ represents the Resistance of the Base B C, $\overline{AF}^2$ will represent the Resistance of the Base F G; therefore $\overline{DF}^2$ will represent the Resistance of the Surface F A G. From $\overline{BE}^2$, the Resistance of the Surface B A C, subtract $\overline{FD}^2$ the Resistance of the Surface F A G, and add $\overline{AF}^2$ the Resistance of the Base F G, and we shall have the Resistance of the Frustum B F G C (when moving in the Direction E A) = $\overline{BE}^2 + \overline{AF}^2 - \overline{FD}^2 = \overline{BE}^2 + \overline{AD}^2$. But $\overline{AD}^2 = \overline{AE}^2 - \overline{ED}^2 = \overline{AE}^2 - 2AE \times ED + \overline{ED}^2$: Therefore the Resistance of the Frustum will be $\overline{BE}^2 + \overline{AE}^2 - 2AE \times ED + \overline{ED}^2 = \overline{AB}^2$

C 2

— 2AE

$- 2AE \times ED + \overline{ED}^2$. This is upon a Supposition that the Resistance of the Base BC is called $\overline{AB}^2$; but if this Resistance, as being by the Supposition a constant Quantity, be called 1, the Resistance of the Frustrum will be $\frac{\overline{AB}^2 - 2AE \times ED + \overline{ED}^2}{\overline{AB}^2}$,

or $1 + \frac{\overline{DE}^2 - 2AE \times ED}{\overline{AB}^2}$: And therefore when

this Quantity (supposing DE and EB constant as in the *Problem*) is a *Maximum* or a *Minimum*, the Resistance of the Frustrum will be so.

Call DE , a ; BE , b ; AE , x ; and we have

$$1 + \frac{\overline{DE}^2 - 2AE \times ED}{\overline{AB}^2} = 1 + \frac{aa - 2ax}{bb + xx}, \text{ whose}$$

$$\text{Fluxion } \frac{2ax\dot{x} - 2aax\dot{x} - 2abb\dot{x}}{xx + bb^2} \text{ (in our Case)}$$

$$= *; \text{ whence } xx - ax - bb = *, \text{ and } x = \frac{1}{2}a \pm \sqrt{\frac{aa}{4} + bb}.$$

But x the Height of the whole Cone must be greater than a , which is but the Height of Part of that Cone; therefore $x = \frac{1}{2}a + \sqrt{\frac{aa}{4} + bb}$; whence we have the following Construction.

Since BC the Diameter of the greater Base, and DE the Height of the Frustrum, are both given; bisect DE in K , draw KC , and produce ED to A , so that KA may be equal to KC ; then will the Triangle ABC represent a Cone, whereof the Frustrum $BFGC$ shall meet with less Resistance,

in

in an uniform Fluid, than any other of the same Base and Altitude. Q. E. I.

That this Resistance is a *Minimum*, and not a *Maximum*, is evident from hence; that the Sign of the Fluxion of this Resistance depends upon the Sign of the Quantity $xx - ax - b b$, which when $x = a$ is negative; and therefore the Resistance of the Frustrum B F G C will be less and less, till the Cone A B C, whereof it is a Frustrum, has obtained the Height already assigned,

COROLL. I.

Therefore the Frustrum B F G C meets with less Resistance than a Cone of the same Base and Altitude.

COROLL. II.

The Resistance of the Frustrum B F G C will be to the Resistance of any other Frustrum B L M C of the same Base and Altitude, as $\overline{NB}^2$ is to $\overline{NB}^2 + \frac{DE}{DA} \times \overline{NA}^2$; supposing N to be the Vertex of the Cone, whereof B L M C is a Frustrum,

Of infinite SERIES.

LEMMA.

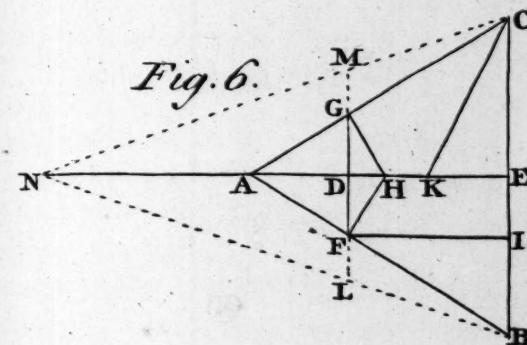
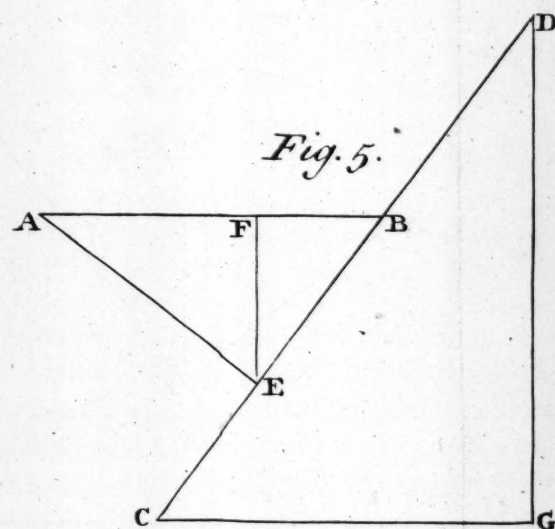
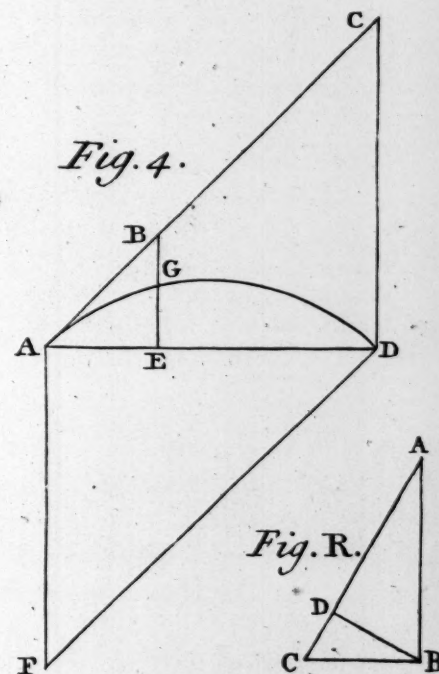
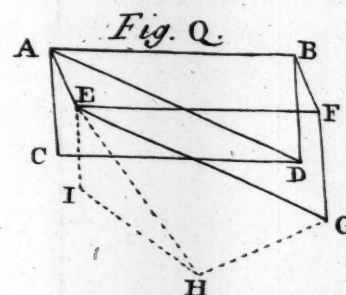
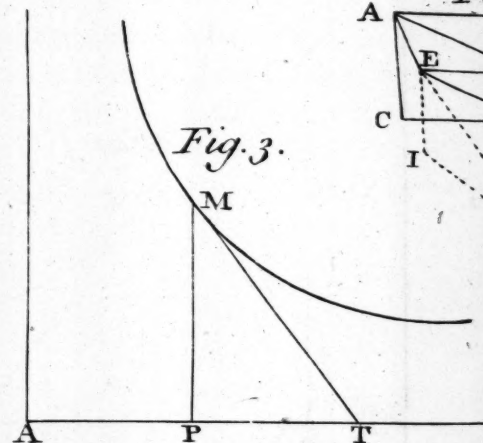
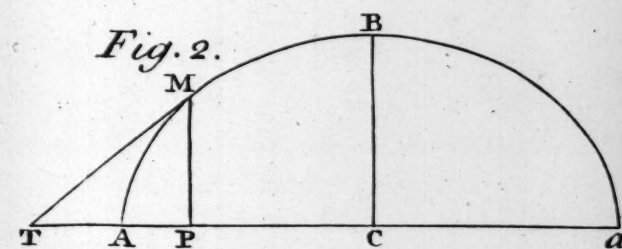
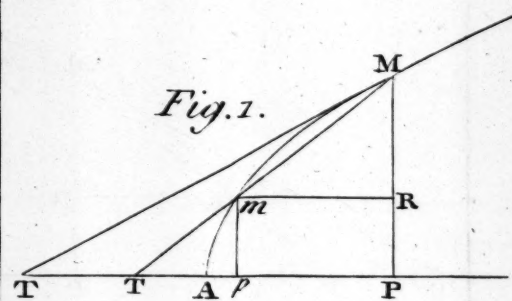
Let $A + Bz + Cz^2, \&c. = a + bz + cz^2, \&c.$ whatever Value is put upon z ; then will $A = a$, $B = b$, $C = c, \&c.$ For since in all Cases $A + Bz + Cz^2 = a + bz + cz^2$, let $z = *$, and we shall have $A = a$: Therefore $Bz + Cz^2 = bz + cz^2$; and $B + Cz = b + cz$ universally. Let z again $= *$, and we have $B = b$; therefore $Cz = cz$, and $C = c, \&c.$ Q. E. D.

THEOREM.

$\overline{p + q} = p^m + \frac{m}{1} \times \frac{Aq}{p} + \frac{m-1}{2} \times \frac{Bq}{p} + \frac{m-2}{3} \times \frac{Cq}{p} + \frac{m-3}{4} \times \frac{Dq}{p} + \frac{m-4}{5} \times \frac{Eq}{p}, \&c.$ where A denotes the first Term p^m , B the second, C the third, $\&c.$

DEMONSTRATION.

Supposing z to be any variable Quantity, let $\overline{p + qz^m} = A + Bz + Cz^2 + Dz^3 + Ez^4, \&c.$ without Restriction: Then supposing $z = *$, we have $p^m = A$; therefore, supposing $z = 1$, we have $\overline{p + q}^m = p^m + B + C + D + E \dots$. Again, since $A + Bz + Cz^2 + Dz^3 + Ez^4 \dots = \overline{p + qz}^n$, we have $Bz + 2Cz^2 + 3Dz^3 + 4Ez^4 \dots = m q z \times \overline{p + qz}^{m-1}$. Therefore $B + 2Cz + 3Dz^2 + 4Ez^3 \dots = (mq \times \overline{p + qz})^{m-1} = mq \times$



$$\begin{aligned} & \times \frac{p+qz^m}{p+qz} = \frac{mqA + mqBz + mqCz^2 + mqDz^3 \text{ \&c.} \dots}{p+qz} \\ & \text{therefore } mqA + mqBz + mqCz^2 + mqDz^3 \dots \\ & = Bp + 2Cp + Bq \times z + 3Dp + 2Cq \times z^2 + \\ & 4Ep + 3Dq \times z^3 \text{ \&c.} \quad \text{Therefore, by the Lemma,} \\ & mqA = Bp; mqB = 2Cp + Bq; mqC = 3Dp + 2Cq; \\ & mqD = 4Ep + 3Dq, \text{ \&c.} \quad \text{which Equations in} \\ & \text{their Order give } \frac{m}{1} \times \frac{Aq}{p} = B, \frac{m-1}{2} \times \frac{Bq}{p} = C, \\ & \frac{m-2}{3} \times \frac{Cq}{p} = D, \frac{m-3}{4} \times \frac{Dq}{p} = E, \text{ \&c.} \quad \text{Therefore} \\ & \frac{p+qz^m}{p+qz} = p^m + \frac{m}{1} \times \frac{Aq}{p} + \frac{m-1}{2} \times \frac{Bq}{p} + \frac{m-2}{3} \times \frac{Cq}{p} + \\ & \frac{m-3}{4} \times \frac{Dq}{p} + \frac{m-4}{5} \times \frac{Eq}{p}, \text{ \&c.} \quad \text{Q. E. D.} \end{aligned}$$

Otherwise thus:

$$\begin{aligned} & \text{Let } \frac{p+qz^m}{p+qz} = A + Bz + Cz^2 + Dz^3 + Ez^4 \dots \\ & \text{as before; then } qz \times \frac{p+qz^m}{p+qz} = Aqz + Bqz^2 \\ & + Cqz^3 + Dqz^4 \text{ \&c.} \quad \text{therefore } \frac{p+qz^{m+1}}{m+1} = \\ & \frac{p^{m+1}}{m+1} + Aqz + \frac{1}{2} Bqz^2 + \frac{1}{3} Cqz^3 + \frac{1}{4} Dqz^4 \text{ \&c.} \\ & \text{as will appear by comparing the Fluents when} \\ & x = *. \quad \text{But } \frac{p+qz^{m+1}}{m+1} = \frac{p+qz}{m+1} \times \frac{p+qz^m}{m+1} = \\ & \frac{p+qz}{m+1} \times \frac{A+Bz+Cz^2+Dz^3+Ez^4 \dots}{m+1} = \\ & \frac{Ap+Bp+qAx+Cp+Bq \times z + Dp+Cq \times z^2 + Ep+Dq \times z^3}{m+1} \end{aligned}$$

C 4

There-

Therefore $\frac{A p + B p + A q \times z + C p + B q \times z^2 + D p + C q \times z^3 + E p + D q \times z^4 \dots}{m+1} = \frac{p^{m+1}}{m+1} + A q z + \frac{1}{2} B q z^2 + \frac{1}{3} C q z^3 + \frac{1}{4} D q z^4, \&c.$ Therefore, by the Lemma, $\frac{A p}{m+1} = \frac{p^{m+1}}{m+1}$; $\frac{B p + A q}{m+1} = A q$; $\frac{C p + B q}{m+1} = \frac{1}{2} B q$; $\frac{D p + C q}{m+1} = \frac{1}{3} C q$; $\frac{E p + D q}{m+1} = \frac{1}{4} D q \&c.$ Therefore, $p^m = A$; $\frac{m}{1} \times \frac{A q}{p} = B$; $\frac{m-1}{2} \times \frac{B q}{p} = C$; $\frac{m-2}{3} \times \frac{C q}{p} = D$; $\frac{m-3}{4} \times \frac{D q}{p} = E \&c.$ as above.

COROLL. I.

$$\overline{p+q}^{\frac{m}{n}} = p^{\frac{m}{n}} + \frac{m}{n} \times \frac{A q}{p} + \frac{m-n}{2n} \frac{B q}{p} + \frac{m-2n}{3n} \frac{C q}{p} + \frac{m-3n}{4n} \frac{D q}{p}, \&c.$$

COROLL. II.

$$r \times \overline{p+q}^{\frac{m}{n}} = r p^{\frac{m}{n}} + \frac{m}{n} \times \frac{A q}{p} + \frac{m-n}{2n} \frac{B q}{p}, \&c.$$

where $A = r p^{\frac{m}{n}} \&c.$

COROLL. III.

If $q = *$, the whole Series will be comprehended in the first Term; therefore the less q is in respect to p , the faster will the Series converge.

Note,

Note. The latter Demonstration of this Proposition cannot well be understood till after the following Problem.

Of the Inverse Method of FLUXIONS.

PROBLEM V.

To find the Fluent of this Fluxion (viz.) $\dot{x} \times x^m$.

SOLUTION.

Suppose the Fluent of this Fluxion to be $p x^q$; then will $p q \dot{x} \times x^{q-1} = \dot{x} \times x^m$: Therefore $p q \times x^{q-1} = 1 \times x^m$; therefore $p q = 1$, and $q - 1 = m$; there-

fore $q = 1 + m$, and $p = \frac{1}{1 + m}$, and the Fluent

sought will be $\frac{1}{1 + m} \times x^{1+m}$.

COROLL.

$\dot{x} \times x^{\frac{m}{n}}$ will have for its Fluent the Quantity $\frac{n}{m+n} \times x^{\frac{m+n}{n}}$.

Quadrature of CURVES.

PROB. VI. Fig. 7.

Let AM be any Curve, whereof AP is the Axis, and MP is a perpendicular Ordinate: It is required to Square, or find the Area of the curvilinear Space APM.

SOLU-

SOLUTION.

Suppose another Ordinate, as mp parallel to and infinitely near MP , and compleat the Parallelograms PQ , pq , RS : Then will the inscribed Parallelogram PR be to the circumscribed one pS as MP to mp ; and consequently in the present Case in a Ratio of Equality: Therefore the Parallelogram PR will be to the curvilinear Space $PMmp$ more, if possible, in a Ratio of Equality; and therefore in the Case where P and p coincide, that Parallelogram and the curvilinear Space may be used for one another.

Call now MP , y ; AP , x ; and we shall have $Pp = \dot{x}$, and the curvilinear Space $PMmp$, or the Parallelogram PR , will be $y\dot{x}$. But this curvilinear Space is the infinitely small Increment or Fluxion of the Area APM : Therefore if the Fluent of this Fluxion be found, which must be had from the known Nature of the Curve, such a Fluent will exhibit the Area sought; provided it begins to flow with the Area: But if it does not, the Excess at the Beginning must be cut off, or the Defect supplied, by the Subtraction or Addition of a known Quantity; and then the Fluent will be equal to the Area sought.

EXAMPLE I.

Let AM be the common *Parabola*, whose Parameter let be p ; and we shall have, from the Nature of the Curve, $px = y^2$, and $y = \sqrt{p} \times \sqrt{x}$,

and $y \dot{x} = \sqrt{p \times \dot{x} \times \sqrt{x}}$, whose Fluent is $\frac{2}{3} \sqrt{p \times x^{\frac{3}{2}}}$, or $\frac{2}{3} \times \sqrt{p \times \sqrt{x} \times x}$, or $\frac{2}{3} x y$: Therefore this Quantity $\frac{2}{3} x y$ flows with the same Velocity with the Area itself. But this Fluent $\frac{2}{3} x y$ is nothing when x is nothing; and the same must be pronounced of the Area A P M: Therefore the Fluent and the Area begin to flow together, and consequently the Area will be equal to the Fluent; *i. e.* the Area A P M will be $\frac{2}{3} A P \times P M$, or two Thirds of the circumscribed Parallelogram P Q.

Note. It may not be improper to observe here, once for all, that if instead of making $p x = y^2$, we had supposed $1 \times x = y^2$, and instead of making A P the Axis, we had supposed it any other Diameter of the Parabola A M, the Proportion between the Area A P M, and the Parallelogram P Q would still have been the same. For however the several Areolæ P M R p would have been changed by such a proportionable Alteration, and the equal Inclination of the several Ordinates M P; the same proportionable Change (and that for the same Reason) would have happened to the Parallelogram P Q.

EXAMPLE II.

Let it now be required to measure the *external Space* A Q M. Since as before $p \times A P = \overline{P M}^2$, or $p \times Q M = \overline{A Q}^2$: If we call A Q, x ; and Q M, y ; we shall have $p y = x x$, and $y = \frac{x x}{p}$, and $y \dot{x} = \frac{x^2 \dot{x}}{p}$,
whose

whose Fluent $\frac{x^3}{3p}$ or $\frac{xy}{3}$ will be the Area sought; so that the external Area A Q M is a third Part of the Parallelogrom P Q; which confirms the former Example.

EXAMPLE III.

Let the Area of the *parabolic Segment* C M T be required.

Since $p \times A B = \overline{B C}^2$, and $p \times A P = \overline{P M}^2 = \overline{B T}^2$; we shall have $p \times B P$, or $p \times M T$, $= \overline{B C}^2 - \overline{B T}^2$; which is equal to the Rectangle C T E. Call now B C, b ; C T, x ; T M, y ; and we shall have $py = 2bx - xx$, and $y = \frac{2bx - xx}{p}$, and $y \dot{x} = \frac{2b\dot{x} - x\dot{x}}{p}$; whose Fluent $\frac{bx - \frac{1}{3}x^3}{p}$ will be the Area of the Segment C M T sought.

Note. When T coincides with B, we shall have $x = BC$, and the parabolic Space C A B will be found equal to $\frac{\overline{B C}^3 - \frac{1}{3}\overline{B C}^3}{p} = \frac{\frac{2}{3}\overline{B C}^3}{p} = \frac{2}{3} A B \times \frac{\overline{B C}^3}{p} = \frac{2}{3} A B \times B C$; which confirms the former Examples.

EXAMPLE IV.

Let the Nature of the *Parabola* be expressed by this general Equation $1 \times x^m = y^n$.

Here

Here we have $y = x_n^{\frac{m}{n}}$, and $y \dot{x} = \dot{x} \times x_n^{\frac{m}{n}}$, whose
 Fluent $\frac{n}{m+n} \times x_n^{\frac{m}{n}+1}$, or $\frac{n}{m+n} x y$ will be the Area
 sought A P M: Therefore the Area A P M will
 be to the Parallelogram P Q as n to $m+n$.

EXAMPLE V. Fig. 8.

Let the Curve X Y be referred to the Line A Z
 by Means of the Ordinate M P parallel to A V,
 both being perpendicular to A Z; and let the
 Nature of the Curve be expressed by the Equation
 $A P \times P M^2 = 1$: Then it is easy to see that the
 Lines A Z and A V will be Asymptotes to the
 Curve; and calling A P, x ; and P M, y ; we shall
 have $y = \frac{1}{\sqrt{x}}$, and $y \dot{x} = \frac{\dot{x}}{\sqrt{x}}$; whose Fluent $2\sqrt{x}$
 is nothing when x is nothing, and infinite when x
 is infinite: which shews that all the Area between
 the infinitely produced Asymptote A V and the
 Ordinate M P, though infinite in Extent, is never-
 theless finite in Quantity, and equal to $2\sqrt{x}$ or $\frac{2x}{\sqrt{x}}$,
 or $2xy$, or twice the Parallelogram P Q. But
 because the Fluent is infinite when x is infinite, it
 shews that all the asymptotic Space on the other
 Side the ordinate M P is infinite as well in Quan-
 tity as in Extent.

EXAMPLE

EXAMPLE VI.

To confirm the foregoing Example, let the same Curve XY be referred to the other Asymptote AV by Means of the Ordinate MQ : Then calling AQ, x ; and MQ, y ; we shall have $y = \frac{1}{x x}$, and $y \dot{x} = \frac{\dot{x}}{x x}$, whose Fluent is $\frac{-1}{x}$. Now this Fluent when x is nothing is $\frac{-1}{0}$, or an infinite negative Quantity; when $x = AQ$, the Fluent is $\frac{-1}{AQ}$; and when x is infinite, the Fluent is nothing. This shews that all the Area between AZ and MQ is infinite, but that all the Area on the other Side MQ is finite, and is equal to $\frac{1}{x}$, or to the Parallelogram PQ , and consequently that all the Area between AV and MP is equal to twice the Parallelogram PQ as before.

EXAMPLE VII.

Let the Nature of the Curve XY when referred to its Asymptote AZ be such that $y = \frac{p}{x^m}$, or $y = p \times x^{-m}$; then will $y \dot{x} = p \dot{x} \times x^{-m}$, whose Fluent $\frac{1}{1-m} \times p x^{1-m}$ will determine the asymptotic Spaces as follows;

1°. If m be less than 1, x^{1-m} will be an affirmative Power of x , and therefore in this Case our
Fluent

Fluent $\frac{1}{1-m} \times p x^{1-m}$ will be nothing when x is nothing; and infinite when x is infinite: therefore, in this Case, all the Area between A V and M P will be finite, and equal to the Parallelogram P Q multiplied into the Quantity $\frac{1}{1-m}$; and therefore this Area, *caeteris paribus*, will be greater or less according to the Quantity $\frac{1}{1-m}$ is so.

2°. If m be greater than 1, x^{1-m} will be a negative Power of x ; therefore in this Case the Fluent $\frac{1}{1-m} \times p x^{1-m}$ will be an infinite Negative when x is infinite: Therefore in this Case the Space between A V and M P will be infinite, and all the Space on the other Side M P will be finite, and equal to the Parallelogram P Q multiplied into the Quantity $\frac{1}{m-1}$; and therefore, *caeteris paribus*, will be greater or less, according as that Quantity $\frac{1}{m-1}$ is so.

3°. From the two foregoing Cases, it follows, that if $m=1$ as in the Case of the common *Hyperbola*, the asymptotic Spaces on both Sides M P will be infinite; for then the Quantities $\frac{1}{1-m}$ or $\frac{1}{m-1}$ will both equal $\frac{1}{\infty}$, and consequently will both be infinite.

EXAMPLE

EXAMPLE VIII. Fig. 9.

Since in the *common Hyperbola* both the asymptotic Spaces are infinite, all that can be expected here is to be able to measure any particular Part of these Spaces. As for Instance: Let MN be a common equilateral Hyperbola, whose Asymptotes are AZ and AV, and let it be required to measure the Space MPQN comprehended between the two perpendicular Ordinates MP, NQ. This may be done various ways, but the most commodious is that which follows:

Bisect PQ in B, and draw the Ordinate BC. Call AB, r ; BP or BQ, x ; and let the Power of the Hyperbola be 1; then will $MP = \frac{1}{r-x}$, and

$NQ = \frac{1}{r+x}$. Let the Ordinates MP, NQ remove into the Places mp , nq infinitely near the former, so that the Space MPNQ may open into the Space $mpqn$: Then it is plain that the Increment or Fluxion of this Space will be

$$MPpm + NQqn = \frac{\dot{x}}{r-x} + \frac{\dot{x}}{r+x} = \frac{2r\dot{x}}{r^2-x^2}. \text{ Now}$$

as this Fluxion is not of the simplest Form, neither can its Fluent be obtained by the simplest Method;

to gain this Fluent the Quantity $\frac{2r\dot{x}}{r^2-x^2}$ must be

thrown into an infinite Series thus; $\frac{2r\dot{x}}{r^2-x^2} = \frac{2\dot{x}}{r} + \frac{2\dot{x}x^2}{r^3} + \frac{2\dot{x}x^4}{r^5} + \frac{2\dot{x}x^6}{r^7} + \frac{2\dot{x}x^8}{r^9} \&c.$ Here it is

plain

plain that the Fluent of every Term of this Series may be found by the simplest Method; so that the

Fluent of the whole will be $\frac{2x}{r} + \frac{2x^3}{3r^3} + \frac{2x^5}{5r^5} + \frac{2x^7}{7r^7} + \frac{2x^9}{9r^9} \&c.$ which Fluent will be equal to the Area sought MPQN. Or thus:

Make $\frac{2x}{r} = A$, $\frac{Ax^2}{r^2} = B$, $\frac{Bx^2}{r^2} = C$, $\frac{Cx^2}{r^2} = D$, $\frac{Dx^2}{r^2} = E \&c.$ and the Area MPQN will be equal to the following infinite Series (*viz.*) $\frac{A}{1} + \frac{B}{3} + \frac{C}{5} + \frac{D}{7} + \frac{E}{9} \&c.$

COROLL. I.

Whatever be the Quantities r and x , so long as they have the same Proportion to one another, the Area MPQN will be the same.

COROLL. II.

Or if the Ordinates MP, NQ be so taken that AQ shall always have the same Proportion to AP, the Area MPQN will always be the same.

For let r and s be two given Quantities, and let AQ and AP be two indeterminate *Abscissae*; but let AQ always be to AP as r to s ; then will $AQ + AP$ be to $AQ - AP$ as $r + s$ to $r - s$. But $AQ + AP = 2AB$, and $AQ - AP = 2BQ$, and $2AB$ is to $2BQ$ as AB to BQ ; therefore AB is to BQ as $r + s$ to $r - s$; therefore the Ratio of AB to BQ will always be the same; therefore,

D

by

by the last Corollary, the Area MPQN will always be the same.

COROLL. III.

If the common equilateral *Hyperbola*, whose *Power* is Unity, be referred to its *Asymptotes*, and if the Difference between any two *Abscissae* be to their Sum as 1 to z ; and lastly, if we make $\frac{2}{z} = A, \frac{A}{zz} = B, \frac{B}{zz} = C, \frac{C}{zz} = D, \frac{D}{zz} = E$: Then will the Area belonging to the Difference of the above-

mentioned *Abscissae* be $A + \frac{B}{3} + \frac{C}{5} + \frac{D}{7} + \frac{E}{9} \&c.$

For in the foregoing Computation the Difference between $r-x$ and $r+x$ is $2x$, and their Sum is $2r$, and $2x$ is to $2r$ as x to r , or as 1 to z ; therefore

$$\frac{x}{r} = \frac{1}{z}, \text{ and } \frac{2x}{r} \text{ or } A = \frac{2}{z}.$$

Of the Computation of Briggs's LOGARITHMS.

THEOREM.

If the common *Hyperbola* be referred to either of its *Asymptotes* by Ordinates parallel to the other; I say that all the *asymptotic Spaces* will be *Logarithms* of the *Abscissae* where they terminate; provided that these *Abscissae* be computed from the Center; and all the *asymptotic Spaces* from one and the same common Ordinate whose *Abscisse* is *Unity*.

Let

Let a and b be the Values of two Abfciffae reckoned both from the Center; and let s be the asymptotic Area reaching from the Abfciffe 1 to the Abfciffe a , and t be a like Area reaching from 1 to b , or from a to ab (since as 1 is to b so is a to ab *); Then since s reaches from 1 to a , and t from a to ab , the Area $s+t$ will reach from the Abfciffe 1 to the Abfciffe whose Value is ab ; therefore the Area s and t are such whose Addition will always answer to the Multiplication of the Abfciffae where they terminate; therefore they are the *Logarithms* of those Abfciffae. Q. E. D.

EXAMPLE.

To compute the hyperbolic Logarithm of 10, that is, to find in the equilateral Hyperbola whose Power is Unity all the asymptotic Area from the Abfciffe 1 to the Abfciffe 10.

If $\frac{2}{3}$ be made equal to A , $\frac{A}{9} = B$, $\frac{B}{9} = C$, $\frac{C}{9} = D$, $\frac{D}{9} = E$ &c. the Area from the Abfciffe 1 to the Abfciffe 2, or the hyperbolic Logarithm of 2 will be $A + \frac{B}{3} + \frac{C}{5} + \frac{D}{7} + \frac{E}{9}$ &c. But the Logarithm of 8 is 3 Times the Logarithm of 2; therefore if instead of making $A = \frac{2}{3}$ we make $A = 2$, and thence derive the other Terms B, C, D, E , &c. as before, we shall have $A + \frac{B}{3} + \frac{C}{5} + \frac{D}{7} + \frac{E}{9}$, &c. the hyperbolic Logarithm of 8. Again, if we

* See Coroll. II. next preceding.

make $\frac{2}{3} = K$, $\frac{K}{81} = L$, $\frac{L}{81} = M$, $\frac{M}{81} = N$, &c.

the hyperbolic Area from 8 to 10 will be $K + \frac{L}{3} + \frac{M}{5} + \frac{N}{7}$ &c. But K or $\frac{2}{3} = B$ in the foregoing Series; therefore L or $\frac{K}{81}$, or $\frac{B}{81}$, or $\frac{C}{9} = D$.

In like Manner $M = F$, $N = H$, &c. and therefore if to the hyperbolic Logarithm of 8 above found be added $B + \frac{D}{3} + \frac{F}{5} + \frac{H}{7}$ &c. we shall have the hyperbolic Logarithm of 10 = 2,302585092994.

*A fuller Computation of the Hyperbolic
LOGARITHM of 10.*

A	2,000000000000000
B	,222222222222222
C	2469135802469
D	274348422497
E	30483158055
F	3387017562
G	376335285
H	41815032
I	4646115
K	516235
L	57359
M	6373
N	708
O	79
P	9
Q	1

A 2,000000000000000

B 7407407407407

C 493827160494

D 39192631785

E 3387017562

F 307910687

G 28948868

H 2787669

I 273301

K 27170

L 2731

M 277

N 28

O 3

27

2,07944154167982 = Log. 8.

D 3

Log.

$\frac{B}{z^2} = C$, &c. the Sum of the Series $A + \frac{B}{3} + \frac{C}{5}$, &c. will be *Briggs's* Logarithm of $\frac{z+1}{z-1}$; for the Sum of this Series will be the Area reaching from the Abscisse $z-1$ to the Abscisse $z+1$, or which is the same Thing, from the Abscisse 1 to the Abscisse whose Value is $\frac{z+1}{z-1}$.

COROLL.

The hyperbolic Logarithm of any Number is to *Briggs's* Logarithm of the same as 1 to 434294481903.

Of the Mensuration of SOLIDS.

PROBLEM VII. Fig. 7.

To find the Content of a Solid generated by the Revolution of any known Curve AM about its Axis AP.

Let the Diameter of a Circle be to the Circumference as 1 to e . Call AP, x ; PM, y ; and let the Ordinate MP make an infinitely small Remove into the Place mp ; then it is plain, that as the Parallelogram PR represented the Fluxion of the Area APM, a Cylinder generated by its Revolution about AP will represent the Fluxion of the Solid generated by the whole Area APM: But the Basis of such a Cylinder is $e yy$, and its Height or Thickness $\dot{x}$. Compute therefore the Fluent

of $eyy\dot{x}$, and that Fluent will determine the Solid, just in the same Manner as the Fluent of $y\dot{x}$ determined the Area.

EXAMPLE I.

Let the Nature of the Curve by whose Area the Solid is generated be expressed by this Equation $p x^m = y^n$, supposing m and n integral and affirmative: Then will $y = p^{\frac{1}{n}} \times x^{\frac{m}{n}}$, and $yy = p^{\frac{2}{n}} \times x^{\frac{2m}{n}}$, and $eyy\dot{x} = e p^{\frac{2}{n}} \times \dot{x} \times x^{\frac{2m}{n}}$; the Fluent whereof is $\frac{n}{2m+n} \times e p^{\frac{2}{n}} \times x^{\frac{2m}{n}+1}$; $= \frac{n}{2m+n} \times eyyx$. But $eyyx$ is a Cylinder having the same Base and Altitude with the Solid proposed: Therefore the Content of the Solid proposed is equal to a Cylinder of the same Base and Altitude multiplied into the Fraction $\frac{n}{2m+n}$.

EXAMPLE II. Fig. 8.

Let it be required to find the Content of the Solid, or rather Part of the Solid, generated by the Revolution of the Curve XY about one of its Asymptotes AZ . Let the Nature of the Curve, when referred to its Asymptote AZ , be $y = \frac{p}{x^m}$ $= p \times x^{-m}$; then will $yy = p^2 \times x^{-2m}$, and $eyy\dot{x} = e p^2 \times x^{-2m} \dot{x}$, whose Fluent $\frac{1}{1-2m} \times e p^2 x^{1-2m}$ shews, That if $2m$ be less than 1, all the Solid generated by the

the Area between A V and M P is finite, and equal to the Cylinder generated by the Parallelogram P Q multiplied into the Fraction $\frac{1}{1-2m}$;

the other Part of the Solid being infinite. On the other hand, if $2m$ be greater than 1, all the Solid between A V and M P will be infinite; and the infinitely extended Solid on the other Side P M will be finite, and equal to the above-mentioned Cylinder multiplied into the Fraction $\frac{1}{2m-1}$.

Of this latter Kind, among many others, is the Solid generated by the *Hyperbola*; which will always be equal to the Cylinder generated by the Parallelogram P Q.

EXAMPLE III. Fig. 10.

To find the Content of a Spheroid generated by the Revolution of the Semi-ellipse A M B about its Axis A P a.

Call C A, t ; C B, c ; C P, x ; M P, y ; and we shall have, from the Nature of the *Ellipse*, $yy = cc - \frac{c^2 x^2}{t^2}$, and $eyy x = eccx - \frac{ec^2 x^3}{3t^2}$, whose Fluent $eccx - \frac{ec^2 x^3}{3t^2}$ is the Content of the Solid generated

by the Area C B M P; therefore if we suppose C P or $x = C A$ or t , we shall have the Hemispheroid generated by the Area C B A $= ecc^2 t - \frac{ec^2 t^3}{3t^2} =$

$\frac{2}{3} ec^2 \times t$: Therefore the solid Content of the whole Spheroid

Spheroid is equal to $\frac{2}{3} ec^2 \times 2t$: But $ec^2 \times 2t$ is the Content of a circumscribing Cylinder.

Of the Surfaces of Curvilinear Solids.

PROB. VIII. Fig. 7.

As the curvilinear Area AMP revolving about its Axis AP generates a Solid, so the Curve AM by such a Revolution generates the curve Surface of that Solid. Let it therefore be required *To find the Surface generated by any known Curve AM revolving about its Axis AP.*

Let the Diameter of a Circle be to the Circumference as 1 to e , and call AP, x ; MP, y ; then will Mm or $\sqrt{MR^2 + Rm^2} = \sqrt{x^2 + y^2}$. But as Mm is the Fluxion of the Curve AM, if AM by its Revolution about AP generates a Surface, Mm will generate the Fluxion of that Surface, which will be an Annulus whose Breadth is Mm , and whose Circumference is equal to the Circumference of a Circle described by the Semidiameter PM: Therefore $2ey\sqrt{x^2 + y^2}$ will be the Fluxion of the Surface; and its Fluent, when found from the Nature of the Curve, will determine the Surface sought.

EXAMPLE I. Fig. 11.

Let AMa be a Semicircle upon the Center C; and let it be required to find the Surface of a Sphere generated by the Revolution of that Semicircle about its Diameter Aa.

Call

to the Menfuration of CURVE SURFACES. 43

Call AC, r ; AP, x ; MP, y ; $Pa, 2r-x$; and we fhall have $yy=2rx-xx$, and $2y\dot{y}=2r\dot{x}-2x\dot{x}$, and $\dot{y} = \frac{r\dot{x}-x\dot{x}}{y}$, $\dot{y}^2 = \frac{r^2\dot{x}^2-2rx\dot{x}^2+x^2\dot{x}^2}{yy} = \frac{r^2\dot{x}^2-y^2\dot{x}^2}{yy} = \frac{r^2\dot{x}^2}{yy} - \dot{x}^2$: Therefore $\dot{x}^2 + \dot{y}^2 = \frac{r^2\dot{x}^2}{yy}$; therefore $yy \times \sqrt{\dot{x}^2 + \dot{y}^2} = r^2 \times \dot{x}$, and $y \times \sqrt{\dot{x}^2 + \dot{y}^2} = r\dot{x}$, and $2ey \times \sqrt{\dot{x}^2 + \dot{y}^2}$ (the Fluxion of the Surface fought) $= e \times 2r\dot{x}$, whose Fluent $e \times 2rx$ will be the Surface of a Segment of the Sphere, whose Axis is $AP (x)$. But from the fimilar Triangles APM , AMa , we have AP to AM as AM to Aa , and confequently $AP \times Aa$, or $2rx = \overline{AM}^2$; therefore $e \times 2rx = e \times \overline{AM}^2$: But $e \times \overline{AM}^2$ is the Area of a Circle whose Radius is AM ; therefore the Surface of any Segment of a Sphere is equal to a Circle whose Semidiameter is a Line drawn from the Pole of the Segment to the Circumference of its Bafe; therefore the Surface of the whole Sphere is equal to a Circle whose Radius is the Diameter of the Sphere, or, which is all one, is equal to four great Circles of the Sphere.

EXAMPLE II. Fig. 12.

Let it be required to eftimate the Surface of the parabolic Solid generated by the common Parabola AM about its Axis AP .

Let F be the Focus of the Parabola, and call AF, m ; AP, x ; MP, y ; and we fhall have $yy=4mx$, and $2y\dot{y}=4m\dot{x}$, and $y\dot{y}=2m\dot{x}$, and $y^2\dot{y}^2$

$y^2 y^2 = 4m^2 x^2$, and $y^2 \times x^2 + y^2 \times y^2$, or $y^2 \times x^2 + y^2$
 $= 4m x x^2 + 4m^2 x^2$, or $4m x + 4m^2 \times x^2$; therefore
 $y \times \sqrt{x^2 + y^2} = x \times \sqrt{4m x + 4m^2}$, and $2e y \times \sqrt{x^2 + y^2}$
 $= 2e x \times \sqrt{4m x + 4m^2}$. Now, to reduce this Fluxion
 $2e x \times \sqrt{4m x + 4m^2}$ into a more convenient Form,
 for the better finding of its Fluent, make $x + m = z$;
 then we shall have $x = z$, and $4m x + 4m^2 = 4m z$,
 and $2e x \times \sqrt{4m x + 4m^2} = 2e z \times \sqrt{4m z}$, whose
 Fluent is $2e \times \frac{2}{3} z \times \sqrt{4m z}$. But at the Beginning
 of the Surface of the parabolic Solid, that is, when
 $x = 0$, this Fluent is $2e \times \frac{2}{3} m \times \sqrt{4m m}$; which Part
 being subtracted from the foregoing Fluent, to
 render it equal to the Surface sought, we shall have
 that Surface equal to $2e \times \frac{2}{3} z \times \sqrt{4m z} - 2e \times \frac{2}{3} m \times$
 $\sqrt{4m m}$.

Now to construct this Fluent, that is, to find
 some geometrical Area with which it may be com-
 pared, set off from the Point F on the Axis con-
 trary to F A the Distance F Q = A P; and through
 F and Q draw the Chords L l, N n, perpendicular
 to the Axis; and since F Q = A P, we shall have
 A Q = A P + A F = x + m = z, and N Q = $\sqrt{4m z}$,
 and $\frac{2}{3} z \times \sqrt{4m z} = \frac{2}{3} A Q \times N Q$ which is equal to the
 parabolic Space A Q N; therefore $2 \times \frac{2}{3} z \times \sqrt{4m z}$
 is equal to the whole Area A N n; and for the
 same Reason $2 \times \frac{2}{3} m \times \sqrt{4m m}$ is equal to the Area
 A L l, and $2 \times \frac{2}{3} z \times \sqrt{4m z} - 2 \times \frac{2}{3} m \times \sqrt{4m m}$ is equal
 to the Area L N n l, and $2e \times \frac{2}{3} z \times \sqrt{4m z} - 2e \times \frac{2}{3} m \times$
 $\sqrt{4m m}$

$\sqrt{4mm}$ is equal to the foresaid Area multiplied into e : Therefore, as the Diameter of a Circle is to the Circumference, so is the Area $LNnl$ to the Surface sought.

PROB. IX. Fig. 13.

It is required having given AB the Radius of any Circle, together with BC the Tangent of any Ark as BD, to find that Ark; and consequently from the given Tangent of some known Ark to find the whole Circumference.

Draw the Secant ADC, and imagine another as $A d c$ infinitely near it, and let CE be perpendicular to $A c$: Then from the similar Triangles $C c E$ and $A c B$, as also $A C E$ and $A D d$, we have the following Proportions; $C c$ to $C E$, as $A c$ or $A C$ to $A B$; or as $\overline{A C}^2$ to $A B \times A C$; and $C E$ to $D d$, as $A C$ to $A D$, or as $A C$ to $A B$, or as $A B \times A C$ to $\overline{A B}^2$; therefore *ex aequo* $C c$ is to $D d$, as $\overline{A C}^2$ to $\overline{A B}^2$.

Now to render the Calculation more simple, call $A B, 1$; $B C, t$; then will $C c = i$, $A C^2 = 1 + t t$,

and $D d = \frac{i}{1 + t t} = i - t t i + t^4 i - t^6 i + t^8 i - \mathcal{E} c$.

whence $B D = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \mathcal{E} c$.

Now to find the whole Circumference, let $B D$ be some known Ark whose Tangent is given. As for Example, let $B D$ be an Ark of 45 Degrees; then will its Tangent $B C = B A$ or 1, and we shall have

$B D$

$BD = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \text{&c.}$ and consequently $8BD$, or the whole Circumference will be 8 Times the foregoing Series. But as this Series, how simple soever, converges too slow for Practice, it will be proper to take a less Ark than that of 45 Degrees, whose Tangent can be computed. As for Instance, let BD be an Ark of 30 Degrees, and produce its Tangent BC to F , till $BF = BC$; then drawing AF it is evident that ACF will be an equilateral Triangle, and consequently that $AC = 2CB$ or $2t$; whence $\overline{AC}^2 (4tt) = \overline{CB}^2 (tt) = AB^2 (1)$, and consequently $tt = \frac{1}{3}$; whence

$$t = \frac{1}{\sqrt{3}}, \quad t^3 = \frac{1}{\sqrt{3} \times 3}, \quad t^5 = \frac{1}{\sqrt{3} \times 9}, \quad t^7 = \frac{1}{\sqrt{3} \times 27},$$

$t^9 = \frac{1}{\sqrt{3} \times 81}$, &c. Therefore in this Case $BD =$

$$\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3} \times 3 \times 3} + \frac{1}{\sqrt{3} \times 9 \times 5} - \frac{1}{\sqrt{3} \times 27 \times 7} + \frac{1}{\sqrt{3} \times 81 \times 9}$$

$$- \text{&c.} = \frac{1}{\sqrt{3}} \times 1 - \frac{1}{3 \times 5} + \frac{1}{9 \times 5} - \frac{1}{27 \times 7} + \frac{1}{81 \times 9}$$

$- \text{&c.}$ Therefore $6BD = \frac{6}{\sqrt{3}}$ multiplied into the

foregoing Series. But $6BD$ is the Semi-circumference of a Circle whose Diameter is 2, or the whole Circumference of a Circle whose Diameter

is 1, and $\frac{6}{\sqrt{3}} = \sqrt{12}$; for if we make $x = \frac{6}{\sqrt{3}}$, we shall have $xx = \frac{36}{3} = 12$, and $x = \sqrt{12}$: There-

fore

fore the Circumference of a Circle whose Diameter

is 1 is equal to $\sqrt{12 \times 1} - \frac{1}{3 \times 3} + \frac{1}{9 \times 5} - \frac{1}{27 \times 7} + \frac{1}{81 \times 9}$

— &c. or if we make $\sqrt{12} = a, \frac{a}{3} = b, \frac{b}{3} = c, \frac{c}{3} = d,$

$\frac{d}{3} = e$, &c. we shall have the Circumference of a

Circle whose Diameter is 1 equal to $a - \frac{1}{3}b + \frac{1}{3}c$

$- \frac{1}{3}d + \frac{1}{3}e$, &c.

See the Work in *Sherwin's Tables*, Page 55.

The Diameter being 1, the Circumference is 3.14159265359.

Of the Method of Reverſing SERIES.

This will beſt be ſeen by the two following Examples.

EXAMPLE I. Fig. 9.

Let MCN be the common *Hyperbola* referred to its *Aſymptote* AB. Let BC be a fixt Ordinate, and NQ a moveable one: It is required, having given the Baſe BQ, to find the Area BCNQ; and *vice verſâ*, having given the Area BCNQ to find the Baſe BQ.

SOLUTION.

Let 1 be the Power of the Hyperbola. Call alſo AB, 1; BQ, x ; NQ, y ; and the Area BCNQ, z ; then we ſhall have $y = \frac{1}{1+x}$, and

$y \dot{x}$ or $\dot{z} = \frac{\dot{x}}{1+x} = \dot{x} - x \dot{x} + x^2 \dot{x} - x^3 \dot{x} + x^4 \dot{x} - \&c.$
Therefore

Therefore $z = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \text{&c.}$
And thus the first Part of the Problem is dispatched.

Let it now be required, having given z , or the Area B C N Q, to find x , or the Base BQ; which is as much as to say, that as we have already expressed the Value of z by an infinite Series wherein the several Powers of x are concerned; it is now required to express the Value of x by an infinite Series wherein the several Powers of z are concerned. In order to which it is necessary that the several Powers of z should first be found, to as many Terms as the Series is intended to be continued. Thus:

$$\text{Equation 1. } z = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$$

$$\text{Eq. 2. } z^2 = x \cdot x - x^2 + \frac{1}{3}x^4 - \frac{1}{2}x^5$$

$$\text{Eq. 3. } z^3 = x \cdot x \cdot x - x^3 + \frac{1}{2}x^4 + \frac{1}{3}x^5$$

$$\text{Eq. 4. } z^4 = x \cdot x \cdot x \cdot x - 2x^5$$

$$\text{Eq. 5. } z^5 = x \cdot x \cdot x \cdot x \cdot x + x^5.$$

Having got these Equations, the whole Business of this Problem is from these Equations to find others, out of which all the Powers of x , except the first shall be expunged successively, so that at last x may be found equal to an infinite Series composed of the several Powers of z . As for Instance. Equation the first is $x - \frac{1}{2}x^2 \text{ &c.} = z$. Now the Term $-\frac{1}{2}x^2$ is to be expunged, by Means of Equation the second, thus: $x^2 - x^3 + \frac{1}{3}x^4 - \frac{1}{2}x^5 = z^2$; therefore $\frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{2 \cdot 3}x^4 - \frac{1}{2 \cdot 2}x^5 = \frac{1}{2}z^2$; which last Equation being added to Equation the first will give

$$\text{Eq. 6. } x \cdot x = \frac{1}{2}x^3 + \frac{1}{2 \cdot 3}x^4 - \frac{1}{2 \cdot 2}x^5 = z + \frac{1}{2}z^2.$$

Here

Here it is easy to see, that the next Term to be expunged is $-\frac{1}{2}x^3$, which is done by Equation 3, thus: $x^3 - \frac{1}{2}x^4 + \frac{1}{4}x^5 = z^3$; therefore $\frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 = \frac{1}{2}z^3$. This last Equation added to Equ. 6 gives

Equ. 7. $x \times x \times x - \frac{1}{24}x^4 + \frac{1}{40}x^5 = z + \frac{1}{2}z^2 + \frac{1}{2}z^3$. The next Term $-\frac{1}{24}x^4$ is expunged by Eq. 4, thus: $x^4 - 2x^5 = z^4$; therefore $\frac{1}{24}x^4 - \frac{1}{12}x^5 = \frac{1}{24}z^4$. This added to Equation 7 gives

Equ. 8. $x \times x \times x \times x - \frac{1}{120}x^5 = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4$. The last Term to be expunged is $-\frac{1}{120}x^5$, which is effected by Equ. 5, thus: $x^5 = z^5$; therefore $\frac{1}{120}x^5 = \frac{1}{120}z^5$, which added to Equ. 8 gives

Equ. 9. $x \times x \times x \times x \times x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$, &c. or thus:

$$x = \frac{z}{1} + \frac{z^2}{1 \times 2} + \frac{z^3}{1 \times 2 \times 3} + \frac{z^4}{1 \times 2 \times 3 \times 4} + \frac{z^5}{1 \times 2 \times 3 \times 4 \times 5}.$$

Sir ISAAC NEWTON'S Theorem for reverfing Series's of the foregoing Kind, viz. where the Quantity is concerned in every Power.

Sit $z = ax + bx^2 + cx^3 + dx^4 + ex^5$ &c. et viciffim erit

$$\begin{aligned} x &= \frac{z}{a} \\ &- \frac{b}{a^3} z^2 \\ &+ \frac{2b^2 - ac}{a^5} z^3 \\ &+ \frac{5abc - 5b^3 - a^2d}{a^7} z^4 \\ &+ \frac{3a^2c^2 - 21a^2b^2c + 6a^2bd + 14b^4 - a^3e}{a^9} z^5, \text{ \&c.} \end{aligned}$$

E

From

From the Solution of the first Part of this Example may be drawn the following useful Observation, (*viz.*) That the Fluent of this Fluxion $\frac{\dot{x}}{x}$ is the hyperbolic Logarithm of x . For if, instead of B Q we call A Q, x ; the Fluxion of the Area B C N Q will be $\frac{\dot{x}}{x}$: Therefore *è converso*, the Fluent of $\frac{\dot{x}}{x}$ is the Area B C N Q, which (supposing A B = 1) is the hyperbolic Logarithm of the Abscisse A Q, as above.

EXAMPLE II. Fig. 14.

It is required, having given A B the Radius of any Circle, together with C D the Sine of any Ark as B C, to find that Ark, and vice versa.

Let $c d$ be an Ordinate infinitely near and parallel to C D, and let $c E$ be parallel to A B. Call A B, 1; C D, y ; B C, z ; C E, $\dot{y}$; C c, $\dot{z}$; and A D, $\sqrt{1-y^2}$: Then from the similar Triangles ADC, C E c we shall have this Proportion. A D ($\sqrt{1-y^2}$) is to A C (1), as C E ($\dot{y}$) is to C c ($\dot{z}$)

$$= \frac{\dot{y}}{\sqrt{1-y^2}} = \dot{y} + \frac{1}{2}\dot{y}y^2 + \frac{3}{8}\dot{y}y^4 + \frac{5}{16}\dot{y}y^6 + \mathcal{E}^2c.$$

Therefore $z = y + \frac{1}{2}y^3 + \frac{3}{8}y^5 + \frac{5}{16}y^7 + \mathcal{E}^2c$. which is the first Part of this Problem.

The other consists in the Computation of the following Equations.

Equ. 1.

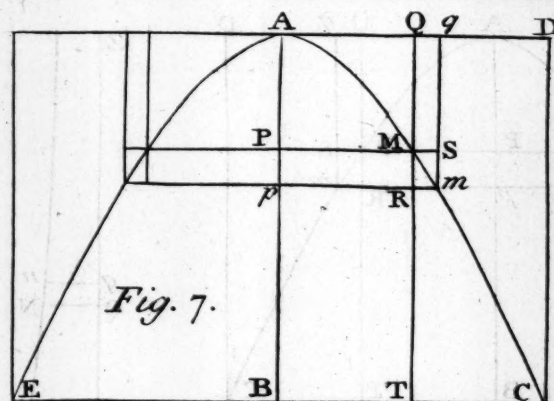


Fig. 7.

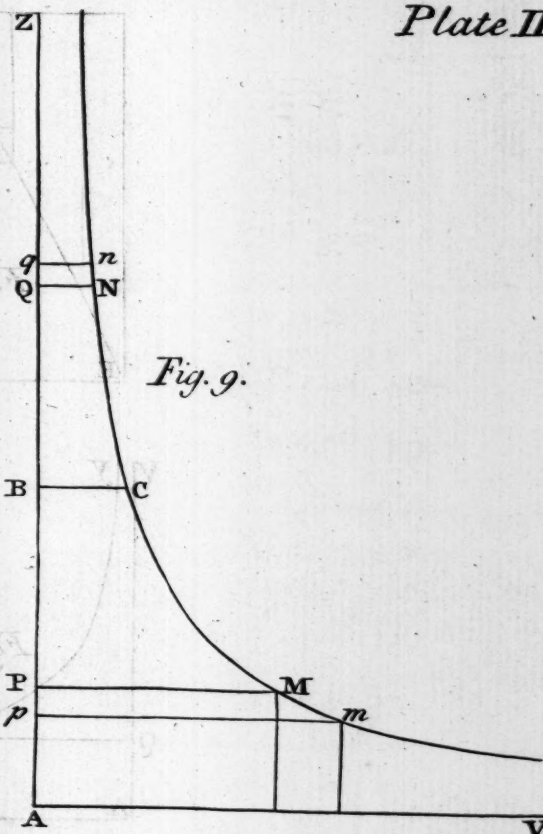


Fig. 9.

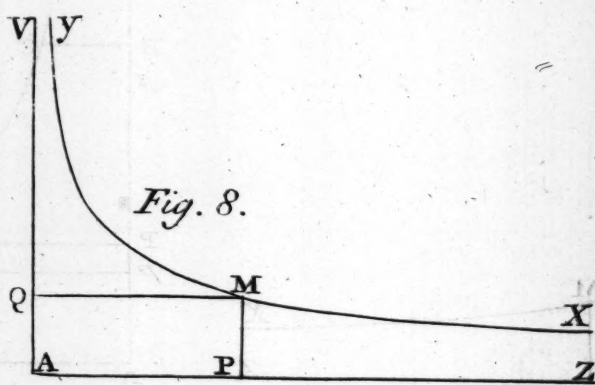


Fig. 8.

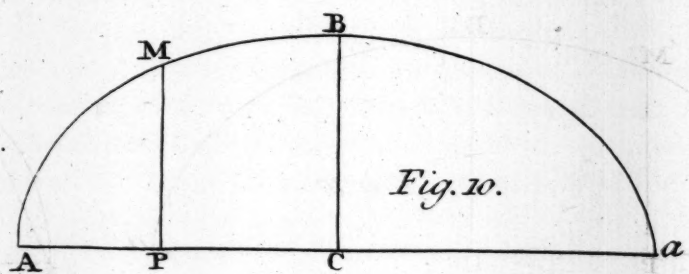


Fig. 10.

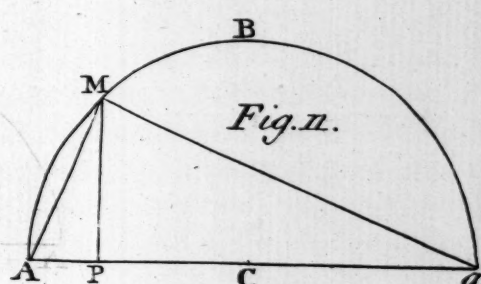


Fig. 11.

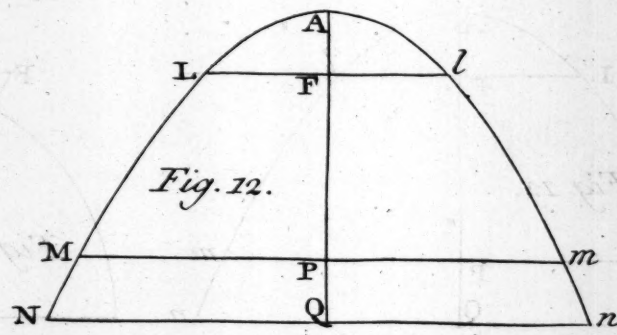


Fig. 12.

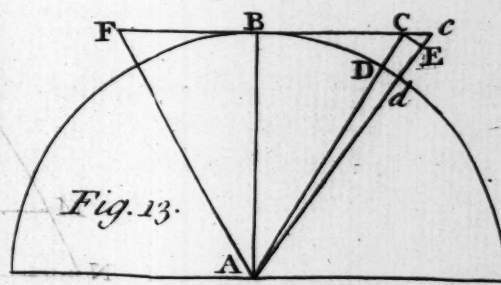


Fig. 13.

$$\begin{aligned}
 \text{Equ. 1. } z &= y + \frac{1}{2}y^3 + \frac{1}{4}y^5 + \frac{1}{112}y^7 \\
 2. \quad z^3 &= * * + y^3 + \frac{1}{2}y^5 + \frac{1}{112}y^7 \\
 3. \quad z^5 &= * * * * + y^5 + \frac{5}{2}y^7 \\
 4. \quad z^7 &= * * * * * + y^7 \\
 5. \quad z - \frac{1}{2}z^3 &= y * - \frac{1}{112}y^5 - \frac{1}{112}y^7 \\
 6. \quad z - \frac{1}{2}z^3 + \frac{1}{112}z^5 &= y * * + \frac{1}{112}y^7 \\
 7. \quad z - \frac{1}{2}z^3 + \frac{1}{112}z^5 - \frac{1}{112}z^7 &+ \text{Ec.} = y.
 \end{aligned}$$

Or thus :

$$y = \frac{z}{1} - \frac{z^3}{1 \times 2 \times 3} + \frac{z^5}{1 \times 2 \times 3 \times 4 \times 5} - \frac{z^7}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} + \text{Ec.}$$

Sir ISAAC NEWTON's Theorem for reversing Series's of this Kind, (viz.) where the Quantity is concerned but in every other Power, beginning with the first Power.

Sit $z = a y + b y^3 + c y^5 + d y^7 + e y^9$, &c. et vicissim erit

$$\begin{aligned}
 y &= \frac{z}{a} \\
 &- \frac{b}{a^4} z^3 \\
 &+ \frac{3b^2 - ac}{a^7} z^5 \\
 &+ \frac{8abc - a^2d - 12b^3}{a^{10}} z^7 \\
 &+ \frac{55b^4 - 55ab^2c + 10a^2bd + 5a^2c^2 - a^3e}{a^{13}} z^9 \\
 &+ \text{Ec.}
 \end{aligned}$$

PROBLEM X.

It is required to find by an infinite Series the Cofine of any proposed Ark in a Circle whose Radius is 1.

Let the Ark proposed be z , and its Sine y ; then will its Cofine be $\sqrt{1-yy}$. But by the last Problem $y = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \mathcal{E}c$. Therefore $yy = z^2 - \frac{1}{3}z^4 + \frac{2}{45}z^6 - \frac{1}{315}z^8 + \mathcal{E}c$. Therefore $1-yy = 1 - z^2 + \frac{1}{3}z^4 - \frac{2}{45}z^6 + \frac{1}{315}z^8 - \mathcal{E}c$. Therefore $\sqrt{1-yy}$ (or the Cofine sought) $= 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \frac{1}{40320}z^8 - \mathcal{E}c$. $= 1 - \frac{z^2}{1 \times 2} + \frac{z^4}{1 \times 2 \times 3 \times 4} - \frac{z^6}{1 \times 2 \times 3 \times 4 \times 5 \times 6} + \frac{z^8}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} - \mathcal{E}c$.

SCHOLIUM.

From the Solution of the last Problem but one it appears, that if y be the right Sine of an Ark of a Circle whose Radius is r , the Fluxion of that

Ark will be $\frac{r\dot{y}}{\sqrt{rr-yy}}$; and *vice versa*, the Fluent of $\frac{r\dot{y}}{\sqrt{rr-yy}}$ is an Ark whose right Sine is y of a

Circle whose Radius is r . And from hence again it follows, that if x be the versed Sine of an Ark of a Circle whose Radius is r , the Fluxion of that

Ark will be $\frac{r\dot{x}}{\sqrt{2rx-xx}}$, and, *vice versa*. For in

this Case $2rx-xx=yy$; therefore $2r\dot{x}-2x\dot{x}=2y\dot{y}$; there-

therefore $\frac{r\dot{x}}{y} = \frac{r\dot{y}}{r-x}$. Substitute $\sqrt{2rx-xx}$ instead of y on one Side, and $\sqrt{rr-yy}$ instead of $r-x$ on the other, and we shall have $\frac{r\dot{x}}{\sqrt{2rx-xx}} = \frac{r\dot{y}}{\sqrt{rr-yy}}$; therefore $\frac{r\dot{x}}{\sqrt{2rx-xx}}$ is the Fluxion of an Ark whose versed Sine is x of a Circle whose Radius is r . Or more briefly thus: In the foregoing Figure, $CD (\sqrt{2rx-xx})$ is to $AC (r)$, as $cE (\dot{x})$ is to $Cc \left(\frac{r\dot{x}}{\sqrt{2rx-xx}} \right)$.

Of the Center of GRAVITY of Bodies.

LEMMA.

Plate III. Fig. 15.

Let a, b, c, d , be any Number of Bodies whose Weights and Places are all given, and let their Centers be placed in the same right Line: it is required to find e their common Center of Gravity.

Let the Line ad in which the Bodies are all supposed to be placed be considered as inflexible, or as a mathematical Lever supported at e , but moveable upon that Center: Then it is plain that as e is the common Center of Gravity of the whole System, and the whole System will be at rest upon that Point; therefore the Sum of the *Momenta* where-with the Bodies a and b endeavour to turn the Lever one Way will be equal to the Sum of the

E 3

Momenta

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Momenta wherewith the Bodies c and d endeavour to turn it the other Way. Let $a \times e a$ represent the *Momentum* wherewith the Body a endeavours to turn the Lever on the Side of a , and the *Momenta* of the rest will be $b \times e b$, $c \times e c$, $d \times e d$, and we shall have this Equation $a \times e a + b \times e b = c \times e c + d \times e d$. Assume now some Point as s any where in the Line $a d$ produced, that the Bodies may lie all on the same Side of s ; then we shall have $e a = s e - s a$, $e b = s e - s b$, $e c = s e - s c$, $e d = s e - s d$. Substitute these new Values instead of the old ones in the former Equation, and it will be transformed into this $a \times s e - a \times s a + b \times s e - b \times s b = c \times s c - c \times s e + d \times s d - d \times s e$; which Equation being resolved gives $s e = \frac{a \times s a + b \times s b + c \times s c + d \times s d}{a + b + c + d}$.

In Words thus. Multiply the Weight of every Body into its Distance from the Points s , divide the Sum of the Products by the Sum of the Weights, and the Quotient will be the Distance of the common Center of Gravity from the same Point s . Q. E. I.

COROLL.

Hence if any Number of Bodies a, b, c, d , be ranged upon a Lever, they will have the same Effect upon it as if all their Weights were accumulated into e their common Center of Gravity. For if we suppose the Lever $s d$ now to turn upon s , the Sum of their *Momenta* in the former Case (I mean when placed at a Distance one from another)

other) will be $a \times sa + b \times sb + c \times sc + d \times sd$; and their Momenta when placed at e will be $\overline{a + b + c + d} \times se$; and these two Sums were found equal in the Demonstration of the foregoing Lemma.

SCHOLIUM.

If the Point s be taken any where among the Bodies, and not in the Line ad produced; then must the Sum of the Products on one Side or the other be considered as affirmative, and consequently the Sum of the Products on the other Side will be negative. Add these together according to the Method used for adding affirmative and negative Quantities together, and the Center of Gravity will be found to lie on the affirmative or negative Side, according as the whole Sum was affirmative or negative. If the whole Sum be nothing, the Point s will be the common Center of Gravity.

INSTANCE.

There is a certain cylindrical Rod 36 Inches long, and weighing one Pound; at whose two Extremities are suspended two Weights, the one of five Pounds, the other of seven: I demand the common Center of Gravity of the Rod and these two Weights.

The Momentum of the Rod is equal to the Momentum of 1 Pound at the Center; (by the *Coroll.*) whence supposing s at the Place of the 7 Pound Weight, and working as the *Lemma* directs,

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we

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we shall have the Distance of the Center of Gravity from that Weight equal to $15\frac{1}{3}$ Inches.

P R O B. X. Fig. 16.

To find the Center of Gravity of any curvilinear Space A B C that hath an Axis, and whose Parts may be supposed to be endued with Gravity. To find also the Center of Gravity of a Solid generated by the Revolution of such a Space about its Axis.

Imagine the whole Space A B C to be resolved into an infinite Number of infinitely narrow Trapezia, such as M N n m, whose proper Centers of Gravity are all in the Axis A D, and let all the Chords M P m, N Q n be at first supposed perpendicular to the Axis. Call A P, x ; M P, y ; P Q, $\dot{x}$; and supposing the Weights of the several Trapezia M N n m, as represented by their Areae, to be multiplied into their respective Distances from the Point A, call the Sum of the Products S; and call the whole Space A B C, representing the Sum of the Weights of all its Parts, s ; and we shall have $\dot{S} = 2y x \dot{x}$, and $\dot{s} = 2y \dot{x}$; therefore the Fluxions of these Fluxions being found will determine the Quantities S and s ; and consequently $\frac{S}{s}$, or the Distance of the Center of Gravity upon the Axis from the Vertex A.

Let now the several Chords M m, N n be supposed to turn upon their respective Points P, Q, so as to become inclined to the Axis, but still parallel

parallel to one another ; and the Center of Gravity of the whole Space A B C will still continue in the same Place : For though by this Means, the Weights of all the Parts be diminished, they are diminished in the same Proportion.

Lastly, Let all the Chords M m, N n, &c. return again into their perpendicular Position, and let the Space A B C generate a Solid about its Axis A D, and if we suppose the Diameter of a Circle to be to the Circumference as 1 to e , and use the Quantities S and s as before, we shall have in this Case $\dot{s} = e y y \dot{x}$, and $\dot{s} = e y y \dot{x}$; whence S and s may be determined as before.

EXAMPLE I.

Let the Nature of the Curve be expressed by this Equation $p \times x^m = y^n$; then we shall have $y = p^{\frac{1}{n}} \times x^{\frac{m}{n}}$, and $2 y \dot{x}$ or $\dot{s} = 2 p^{\frac{1}{n}} \times x^{\frac{m}{n}} \times \dot{x}$; whence $S = \frac{2n}{m+2n} \times p^{\frac{1}{n}} \times x^{\frac{m+2n}{n}}$, or $\frac{2n x}{m+2n} \times x y$. Moreover $\dot{s}$ or $2 y \dot{x} = 2 p^{\frac{1}{n}} \times x^{\frac{m}{n}} \times \dot{x}$; therefore $s = \frac{2n}{m+n} \times x y$; therefore $\frac{S}{s} = x \times \frac{m+n}{m+2n}$; therefore the Distance of the Center of Gravity of the whole Space A B C from the Vertex A is $A D \times \frac{m+n}{m+2n}$.

EXAMPLE II.

For the Center of Gravity of the *Solid* we shall have according to the foregoing Notation, $e y y \dot{x}$
or

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or $\dot{s} = e p^{\frac{2}{n}} \times x^{\frac{2m+n}{n}} \times \dot{x}$; therefore $S = \frac{n}{2m+n}$
 $e p^{\frac{2}{n}} \times x^{\frac{2m+n}{n}} = \frac{n x}{2m+n} \times e y^2 x$. Again; or $e y y \dot{x}$
 $= e p^{\frac{2}{n}} \times x^{\frac{2m}{n}} \times \dot{x}$; therefore $s = \frac{n}{2m+n} \times e y^2 x$; there-
fore $\frac{S}{s} = \frac{2m+n}{2n+n} \times x$; and the Distance of the Center
of Gravity of the whole Solid will be the Axis A D
multiplied into the Fraction $\frac{2m+n}{2m+2n}$.

SCHOLIUM.

As the two former Examples will give a Solution to this Problem in an infinite Number of curvilinear Figures, so will it also in Triangles and Parallelograms, and the Solids generated by their Revolutions about their Axis, (*viz.*) Cones and Cylinders. For in a Triangle $p x = y$, and in a Parallelogram $p = y$, or $p x^0 = y^1$.

THEOREM. Fig. 18.

If in a Triangle as A B C, and from any Angle as A be drawn a Line as A E bisecting the opposite Side; and if upon that Line be set off $A F = \frac{2}{3} A E$, the Point F will be the Center of Gravity of the Triangle.

For as the Line A E bisects the Line B C, it must bisect every other Line parallel to it; and consequently the Center of Gravity must be somewhere in that Line A E. For the same Reason if B D be drawn bisecting A C, the Center of Gravity must

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must also be somewhere in the Line BD; therefore it must be in the common Interfection F. Join DE, which as it bisects both AC and BC will be parallel to AB; and we shall have, from similar Triangles, AF to FE as AB to DE, or as BC to BE, or as 2 to 1; and (*componendo*) AF to AE as 2 to 3. Q. E. D.

EXAMPLE III. Fig. 17.

To find the Center of Gravity of an Hemisphere generated by the Revolution of a Semicircle BAC about its Axis AD.

Here it will be proper, for a more simple Calculation, to call DP, x ; and MP, y ; as before: and if we call the Radius r ; we shall have $yy = rr - xx$, and $eyyx\dot{x}$, or $\dot{S} = er r x \dot{x} - ex^3 \dot{x}$; therefore in the Case of the Solid generated by the

Area MB C m, we have $S = \frac{er r x x}{2} - \frac{ex^4}{4}$, and in

the Case of the whole Hemisphere $S = \frac{er^4}{2} - \frac{er^4}{4} = \frac{er^4}{4}$.

Again; or $eyy\dot{x} = er^2\dot{x} - ex^2\dot{x}$; therefore $s = er^2x - \frac{1}{3}ex^3$; which in the Case of the whole Hemisphere is equal to $er^3 - \frac{1}{3}er^3 = \frac{2}{3}er^3$; therefore

$\frac{S}{s} = \frac{3}{4}r$: That is, in an Hemisphere, or Hemispheroid, the Center of Gravity is $\frac{3}{4}$ of the whole Axis from the Base. For if all the elementary Laminæ be contracted in a given Proportion, they will form an Hemispheroid.

EXAMPLE

EXAMPLE IV.

Let us in the next Place enquire into the Center of Gravity of the generating Semicircle.

And here since $yy = rr - xx$, we shall have $2x\dot{x} = -2y\dot{y}$, and $2yx\dot{x}$ or $\dot{S} = -2y\dot{y}$, whose Fluent $-\frac{2y^3}{3}$ when $x=0$ will be $-\frac{2r^3}{3}$, and when $x=r$ will be 0; therefore $S = \frac{2r^3}{3}$: But s is the Area of the Semicircle $ABC = \frac{err}{3}$; therefore $\frac{S}{s} = \frac{4r}{3e}$; which is as much as to say, that as 3 Times the Circumference of any Circle is to 4 Times the Diameter (or as 33 to 14 very nearly) so is the whole Axis of any Semicircle, or Semi-ellipse, to the Part between the Base and the Center of Gravity.

Two PROBLEMS concerning GRAVITY.

PROBLEM XI.

To find the Weight of an infinitely tall Prism or Cylinder standing upon the Surface of the Earth, upon a Supposition that the Force of Gravity in all Places is reciprocally as the Squares of their Distances from the Center of the Earth.

Let the Cylinder in question be distinguished into an infinite Number of infinitely thin *Laminae*, and let x be the Thickness of any one of these *Laminae* at any indeterminate Altitude x from the Center

Center of the Earth; Then will its Weight be as $\frac{\dot{x}}{x x}$ (*i. e.*) $\frac{\dot{x}}{x x}$ will be the Fluxion of the Weight of the Cylinder; whose Fluent is $-\frac{1}{x}$. Call the Se-

midiameter of the Earth r ; and the Value of this Fluent at the Earth's Surface will be $-\frac{1}{r}$, and at an infinite Height it will be nothing; therefore the Weight of this infinite Cylinder will be $\frac{1}{r}$.

Let p be the Weight of a small Part of this Cylinder, at the Surface of the Earth, whose Altitude is a ; and its Weight, according the foregoing Proportion, ought to be represented by $\frac{a}{r r}$; therefore the Weight p will be to the Weight of the infinitely tall Cylinder, as $\frac{a}{r r}$ to $\frac{1}{r}$, *i. e.* as a to r .

Q. E. I.

PROBLEM XII.

Plate IV. Fig. 21.

Let S be the Center of the Earth, B any Point in its Surface, and let the Force of Gravity in all Places be reciprocally as the Squares of their Distances from the Center of the Earth: *It is required to determine the Velocity of a heavy Body at the Surface of the Earth, which it acquires in falling from any given Altitude A B.*

Let x be any indeterminate Distance from the Center of the Earth, and let v be the Velocity of the
the

the falling Body at that Distance. Let $\frac{1}{SB^2}$ represent the Force of Gravity at B, and consequently $\frac{1}{xx}$ its Force at the Distance x . First then it is plain that after the falling Body is arrived at the Distance x , and then descends further through any infinitely small Space as $\dot{x}$, the Time of that infinitely small Descent will be as $\frac{\dot{x}}{v}$, that is, it will be as the Space directly, and as the Velocity inversely; and the infinitely small Acquisition made by the Velocity in that Descent will be as that Time $\frac{\dot{x}}{v}$ and the accelerating Force $\frac{1}{xx}$ jointly; that is, $\dot{v}$ will be as $\frac{\dot{x}}{vx^2}$; therefore $v \dot{v}$ will be as $\frac{\dot{x}}{xx}$; therefore the Fluents of these Fluxions that are generated in equal Times will be proportionable, that is, $\frac{1}{2} v v$ will be as $\frac{1}{SB} - \frac{1}{SA}$, or as $\frac{SA - SB}{SB \times SA}$, or as $\frac{AB}{SB \times SA}$: Therefore, because of the constant Quantities 2 and SB, vv will be as $\frac{AB}{AS}$.

This being discovered, let DB be the Height from whence a Body will fall to the Surface of the Earth in 1 *Second* of Time; and since during so small a Descent, the Force of Gravity may be looked upon as uniform, it is evident that a Body

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falling

falling from D to B will acquire a Velocity which will carry it uniformly through the Space 2BD in a *Second* of Time. Let 2BD represent this Velocity; then must every other Velocity be represented by the Space through which it will carry a Body in a Second of Time. Now to find the Velocity acquired in falling from A to B, I say, as $\frac{DB}{DS}$ is to $\frac{AB}{AS}$, so is $4DB^2$ (the Square of the Velocity acquired in falling from D to B) to $4DB \times DS \times \frac{AB}{AS}$ (the Square of the Velocity acquired in falling from A to B) = $4m \times \frac{AB}{AS}$, supposing m to be a mean Proportional between DB and DS: Therefore a Body falling from B to A acquires a Velocity that would carry it through the Space $2m \times \sqrt{\frac{AB}{AS}}$ in a *Second* of Time. Q. E. I.

SCHOLIUM.

DB is found by Experiment to be $15\frac{1}{12}$ Paris Feet, and DS $19615815\frac{1}{12}$ of the same Measure; therefore $2m$ is about seven *English* Miles.

COROLL. I.

If AB be infinite, the Quantity $\frac{AB}{AS}$ becomes equal to 1, and therefore goes out of the Question; and therefore a Body falling from an infinite Height will acquire at the Surface of the Earth but a finite Velocity,

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Velocity, (*viz.*) such a Velocity as will carry a moving Body uniformly through 7 Miles in 1 Second of Time.

COROLL. II.

Therefore, *è converſo*, if a Body be thrown upwards with ſuch a Velocity, it will never return, but aſcend *ad infinitum*.

COROLL. III.

After the ſame Manner may be determined the Velocity of a falling Body, whatever be the Law of Gravity.

Of the MOTION of PENDULUMS.

LEMMA.

Plate III. Fig. 19.

If two Bodies fall from equal Altitudes, their Velocities at all other equal Altitudes will be equal, though one deſcends obliquely, and the other perpendicularly.

DEMONSTRATION.

Let ABC be an horizontal Chord of any Curve ADC whoſe loweſt Point is D , and whoſe Axis BD is perpendicular to the Horizon. Let E be any Point in the Curve AD , and e another infinitely near it, and above it, and draw horizontal Chords EE and ee cutting the Axis BD in F and f reſpectively: Then imagine a Body as p to ſlide from reſt at A along the Curve AD , and another

other as q to fall perpendicularly from a Point H in the Axis BD ; and let the Point H be so taken, that the Velocity of the Body q in the Point f acquired in falling from H to f may be precisely equal to the Velocity of the Body p in the Point e acquired in falling obliquely from A to e : Then it is plain that the Time wherein the infinitely small Space Ee is described will be to the Time the Space Ff is described in as Ee to Ff ; and if Ee be considered as an inclined Plane, it is further evident, from the Nature of such an inclined Plane, that the Force wherewith the Motion of the Body p is accelerated in falling from e to E is to the Force whereby the Body q is accelerated in falling directly from f to F as Ff is to Ee . Therefore the Time Ee is described in, is as much greater than the Time Ff is described in as the Vis acceleratrix in f is greater than the Vis acceleratrix in e : Therefore the Increments gained to the Velocities of the Bodies p and q in falling from e to E , and from f to F (which are as the Times and the respective Forces considered together) will be equal: Therefore the Velocities of the Bodies p and q in the Points E and F will still be equal; which is as much as to say, that if the Velocities of the two Bodies p and q falling from A and H be in any one particular Case equal at equal Altitudes, they will always be so: But at the Point A the Velocity of the Body p was 0; therefore at the Point B the Velocity of the Body q was 0; therefore the Point H coincides with the

F

Point

Point B: Therefore if two Bodies fall from equal Altitudes, their Velocities at all other equal Altitudes will be equal, though one descends obliquely, and the other perpendicularly. Q. E. D.

COROLL. I.

The Velocity of the Body *p* in the Point E will be in a subduplicate Ratio of BF the Distance between the two horizontal Lines AB and EF, because the Velocity of the Body *q* in the Point F is so.

COROLL. II.

If the Ark ADC be Part of a Circle whose Diameter is DI, the Velocity of a Body in the lowest Point D, acquired in falling through the Ark AD, will be to the Velocity of a Body in the same Point by falling through the Ark ED, as the Chord of the Ark AD is to the Chord of the Ark ED; for the former Velocity is to the latter as $\sqrt{DB}$ is to $\sqrt{DF}$, or as $\sqrt{DI \times DB}$ is to $\sqrt{DI \times DF}$. But $\sqrt{DI \times DB}$ is the Chord of the Ark AD, and $\sqrt{DI \times DF}$ is the Chord of the Ark ED.

P R O-

PROBLEM XIV.

To find precisely the Time of a least Oscillation of a given Pendulum swinging in an Ark of a Circle; and to find, without any sensible Error, also the Time of any other.

Retaining the Construction of the last Lemma; and supposing moreover that the Curve A D C is an Ark of a Circle whose Diameter is D I, and wherein the Pendulum performs its several Oscillations, being now supposed to be at the Point E in its Ascent from D to C. Let $\sqrt{B F}$ express the Velocity acquired by a heavy Body in falling from B to F, and consequently the Velocity of the Pendulum at the Point E; then will $\frac{1}{2}\sqrt{I D}$ express the Velocity acquired in falling through a Space equal to $\frac{1}{2} I D$: And since a Body by this Velocity would be carried uniformly through a Space equal to $\frac{1}{2} I D$ in the same Time as it would fall through $\frac{1}{2} I D$; divide the Space $\frac{1}{2} I D$ by the Velocity $\frac{1}{2}\sqrt{I D}$, and let the Quotient $\sqrt{I D}$ express the Time wherein a heavy Body would fall through $\frac{1}{2} I D$, i. e. through half the Length of

the Pendulum; then will $\frac{E e}{\sqrt{B F}}$ express the Time wherein the Ark E e is described by the Pendulum. But the Ark E e is the Fluxion or Increment of the Ark D E, and (by the Scholium to the tenth

Problem) is equal to $\frac{\frac{1}{2} ID \times Ff}{\sqrt{IF \times FD}} = \frac{\sqrt{ID}}{IF} \times \sqrt{ID}$
 $\times \frac{1}{2} \sqrt{FD}$: Therefore $\frac{Ee}{\sqrt{BF}} = \frac{\sqrt{ID}}{IF} \times \sqrt{ID}$
 $\times \frac{1}{2} \sqrt{FD}$. Bisect BD in K , and KD in L ,

and if the Ark ADC be not very large, the Quantity IF cannot differ sensibly from its middle Quantity IK , nor $\frac{\sqrt{ID}}{\sqrt{IF}}$ from $\frac{IL}{IK}$; therefore

$\frac{Ee}{\sqrt{BF}}$ is equal very nearly to $\frac{IL}{IK} \times \sqrt{ID} \times \frac{1}{2} \sqrt{FD}$.

Upon the Diameter BD describe a Circle $BGDGB$ cutting the Chords EE and ee in G and g respectively, and $\frac{\frac{1}{2} BD \times Ff}{\sqrt{FB \times FD}}$ will be equal to Gg the

Fluxion of the Ark DG ; therefore $\frac{\frac{1}{2} Ff}{\sqrt{FB \times FD}}$

$= \frac{Gg}{BD}$; therefore $\frac{Ee}{\sqrt{BF}}$, or the Time where-

in the Pendulum ascends from E to e will be

$\frac{IL}{IK} \times \sqrt{ID} \times \frac{Gg}{BD}$: Therefore the whole Time of its

Ascent from D to C will be $\frac{IL}{IK} \times \sqrt{ID} \times \frac{DGB}{BD}$,

and the Time of an entire Oscillation through the

whole Ark ADC will be $\frac{IL}{IK} \times \sqrt{ID} \times \frac{BGDGB}{BD}$.

Let the Ark ADC be infinitely small and the

the Quantity $\frac{IL}{IK}$ will become equal to Unity, and

the

the Time of the least Oscillation will be $\sqrt{ID}$

$\times \frac{BGDGB}{BD}$: That is, As the Diameter of any

Circle is to the Circumference, so is $(\sqrt{ID})$ the Time of a heavy Body's falling from Rest through half the Length of a Pendulum, to the Time of a least Oscillation of that Pendulum. Call this Time thus found t , and if the Ark ADC be not very large, the Time of an Oscillation through

that Ark will be $t \times \frac{IL}{IK}$, or it will be $t + t \times \frac{KL}{IK}$.

Q. E. I.

SCHOLIUM.

If the Ark ADC were a Semicircle, the Time of a least Vibration would be to the Time of a Vibration through that Semicircle according to the Proportion above laid down as 6 to 7, or as 29 to $33\frac{1}{2}$; and by other Means we know that the Consequent of this last Proportion ought to be less than $34\frac{1}{2}$, and greater than $34\frac{1}{3}$. If therefore the Proportion above found approaches so near the Truth in so large an Ark as a Semicircle, and is absolutely true in an Ark infinitely small, it will be easy to conjecture how inconsiderable the Error will be in small Arks such as are usually described by Pendulums.

COROLL. I.

If two pendulous Boxes of the same Figure and Magnitude be, together with their Contents, of the

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same

same Weight, and at equal Distances from their Points of Suspension perform their Oscillations in the same Time; it is an Argument that the Quantities of Matter of both these Pendulums are the same, let the Parts be ever so heterogeneous; and consequently that the Matter in those Bodies will be proportionable to their Weights: For if these two Pendulums were to be let fall from Rest, they would fall through half their Lengths in equal Times, and consequently would be equally accelerated.

COROLL. II.

The Space through which a heavy Body falls from Rest in any given Time is to half the Length of a Pendulum that oscillates in that Time, as the Square of the Circumference of any Circle is to the Square of the Diameter: For as the two first in the Proportion are Spaces, the two last represent the Squares of the Times wherein those Spaces are described.

COROLL. III.

Small Vibrations of the same Pendulum are all performed in the same Time nearly: For the small Quantity of Time $t \times \frac{KL}{IK}$ which is to be added to t the Time of a least Oscillation will (in this Case) cause no sensible Variation in the whole.

COROLL.

COROLL. IV.

The Length of any Pendulum is as the Square of the Time of a least Oscillation of that Pendulum multiplied into the absolute Force of Gravity; For if a heavy Body falls through any Space as l in any Time, that Space l will be as the Square of the Time multiplied into the Force of Gravity, But the Time of Descent through l is as the Time of Descent through $\frac{1}{2}l$, and this again is as the Time of a least Oscillation of a Pendulum whose Length is l ; for the same Reason that the Circumference of a Circle is as the Diameter; Therefore the Length of any Pendulum is as the Square of the Time of a least Oscillation of that Pendulum multiplied into the absolute Force of Gravity.

N. B. If this Corollary be applied to the Oscillation of a Pendulum upon an inclined Plane, instead of the absolute Force of Gravity, understand that Force wherewith the Body endeavours to descend in a right Line along that Plane.

COROLL. V.

Therefore the Time of a least Oscillation of any Pendulum is in a subduplicate Ratio of the Length of the Pendulum directly, and in a subduplicate Ratio of the Force of Gravity inversely. For if v be the Force of Gravity, l the Length of any Pendulum, and t the Time of a least Oscillation; $v t^2$ will be as l , by the last Corollary; therefore

t will be as $\sqrt{\frac{l}{v}}$

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COROLL.

COROLL. VI.

Therefore, if the Time of a least Oscillation be given, $\sqrt{\frac{l}{v}}$, and consequently $\frac{l}{v}$, will be a given Quantity, in which v will be as l ; that is, the Force of Gravity will be every where as the Length of an isochronous Pendulum.

COROLL. VII.

If the Force of Gravity be given, the Time of a least Oscillation will be in the subduplicate Ratio of the Length of the Pendulum.

COROLL. VIII.

Universally, If v be the Force of Gravity, l the Length of any Pendulum, n any Number of Vibrations of that Pendulum, and t the Time wherein that Number of Vibrations is performed; the Quantity $v t^2$ will always be as the Quantity $l n^2$: For if $\sqrt{\frac{l}{v}}$ represents the Time of a single Oscillation; then $\sqrt{\frac{l}{v}} \times n$ will represent the Time of any Number n of those Vibrations, that is, the Time t will be as $\sqrt{\frac{l}{v}} \times n$; therefore t^2 will be as $\frac{l n^2}{v}$, and $v t^2$ as $l n^2$.

COROLL.

COROLL. IX.

The Times of similar Oscillations (by which is meant such as are made in similar Arks) will always be in a subduplicate Ratio of the Length of the Pendulums, whatever be the Quantities of those Arks, provided the Force of Gravity continues the same. For if the Arks AD, ED, *e*D, and consequently E*e* be as the Diameter DI, the Lines BF and B*f* will be so too; and $\frac{Ee}{\sqrt{BF}}$, or the Time wherein the Ark E*e* is described will be as $\frac{DI}{\sqrt{DI}}$, *i. e.* as $\sqrt{DI}$, and (*componendo*) the whole Time of an Oscillation through the whole Ark ADC will be as $\sqrt{DI}$, or as $\frac{\sqrt{DI}}{2}$, that is, in a subduplicate Ratio of the Length of the Pendulum.

COROLL. X.

The Force wherewith any Pendulum whose Length is *l* is accelerated in any Point E of its Ark will be as $\frac{EF}{l}$: For if the Weight of the Pendulum, or the absolute Force of Gravity, be called 1, the accelerating Force at E will be $\frac{Ff}{Ee}$: But $Ee = \frac{1}{2} \frac{DI \times Ff}{\sqrt{IF \times FD}} = l \times \frac{Ff}{EF}$: Therefore $\frac{Ee}{Ff} = \frac{l}{EF}$, and $\frac{Ff}{Ee} = \frac{EF}{l}$.

COROLL.

COROLL. XI.

Therefore in small Oscillations, the Force wherewith the same Pendulum is accelerated in its Motion will always be as its Distance from the lowest Point of its Ark; for in this Case the Sine EF differs nothing from the Ark ED .

COROLL. XII.

If the Ark ADC be supposed infinitely small, the Velocity of the same Pendulum at any Point as E will be as $\sqrt{DA^2 - DE^2}$; for this Velocity will always be as $\sqrt{BF}$, that is, as $\sqrt{BD - FD}$, that is, in the same Circle as $\sqrt{BD \times DI - FD \times DI}$. But $BD \times DI$ is the Square of the Chord of the Ark AD , and therefore in our Case $BD \times DI$ is the Square of the Ark AD ; and for the same Reason $FD \times DI$ is the Square of the Ark ED ; Therefore the Velocity of the Pendulum in the Point E is as $\sqrt{DA^2 - DE^2}$.

N. B. The following *Theorem* with its Corollaries are added for the better understanding of what is delivered in the *Principia* concerning the Motion of Fluids, and of Sounds; and that without the help of the *Cycloid*.

THEOREM.

Fig. 20.

If any Body as P be made to oscillate in a strait Line ADC , whose middle Point is D , being accelerated

celerated in its Motion from A to D, and equally retarded in its Motion from D to C by Forces every where proportionable to the Distance of the Body from the middle Point D; I say then that the Body P will perform its Oscillations in the same Time, whatever be the Length of the Line ADC.

To demonstrate this, imagine two such Bodies P and *p* beginning their Oscillations at the same Time, and both oscillating in the same Line (*viz.*) P in a large Space ADC, and *p* in a more contracted Space *aDc*: Then it is plain, that the Accelerations and Velocities of these two Bodies, and consequently the Spaces passed through in any given Time will always be in the same Proportion (*viz.*) in the Proportion of DA to *Da*: Therefore the Bodies P and *p* will both come to the Point D at the same Time, and consequently both perform their Oscillations in the same Time.

For the better conceiving this, imagine the Force whereby the two Bodies P and *p* are agitated to act not continually but by successive finite Impulses at equal Intervals of Time, and always proportionable to the Distances of the Bodies from the Point D; and then imagine those Impulses and Intervals to be diminished *ad infinitum*.

COROLL. I.

Supposing the Body P to oscillate in the Line ADC as above, and to be every where accelerated or retarded by a Force which is to the absolute
Force

Force of Gravity as the Distance DP is to a given Line L , or as $\frac{DP}{L}$ to 1; I say then, that the Time of an Oscillation through that Space ADC will be equal to the Time of a least Oscillation of a Pendulum whose Length is L .

For supposing the Line ADC to be infinitely small, and supposing a Pendulum whose Length is L to swing in an Ark equal to the Line ADC , the Vis acceleratrix of a Body P at any Distance DP will be equal to the Vis acceleratrix of the Pendulum at the same Distance from the lowest Point of its Ark, by the tenth and eleventh Corollaries of the foregoing Proposition: Therefore their Accelerations and Velocities in correspondent Points will be equal, and consequently also the Times of their Oscillations. But the Time of an Oscillation through the Line ADC will always be the same, whatever be the Length of that Line: Therefore universally the Time of an Oscillation through the Space ADC will be equal to the Time of a least Oscillation of a Pendulum whose Length is L .

COROLL. II.

Since the Force whereby the Body P is accelerated or retarded is every where as $\frac{DP}{L}$; and consequently as $\frac{1}{L}$, if the Distance DP be a given Quantity; it follows, that if the accelerating or retarding Forces in every Point of the Line ADC be

be proportionably intended or remitted, the Length L will be reciprocally as the Force at any given Distance. But the Time of a least Oscillation of a Pendulum whose Length is L , and consequently the Time of an Oscillation through the Space ADC , is as $\sqrt{L}$, by the seventh Corollary of the foregoing Proposition: Therefore the Time of an Oscillation through the Space ADC is reciprocally in a subduplicate Ratio of the accelerating or retarding Force at any given Distance from D .

COROLL. III.

By the Demonstration of the foregoing Theorem, it appears, that if a Body as P oscillates in a larger Space ADC , and another as p oscillates in the same Line, but in a more contracted Space aDc , and if the Distances DP , Dp , be taken in the same Proportion as is DA to Da , that then the Velocity in P will be to the Velocity in p as DA to Da : Therefore if the Space ADC be sometimes enlarged, sometimes contracted (the accelerating Forces at the same Distances continuing always the same) and if the Distance DP be always taken in the same Proportion to DA , the Velocity in the Point P will also be as DA .

COROLL. IV.

These Things supposed, and also that the accelerating and retarding Forces at the same Distances from D continue always the same: If upon the Line ADC as a Diameter, be described a Semi-

Semicircle $A M C$, and a Line as $M P$ be drawn any where perpendicular to $A C$ in P ; I say, that the Velocity of the oscillating Body in the Point P will always be as that Perpendicular $P M$: For if we suppose the Line $A P D C$ to be diminished *in infinitum*, the Velocity in any Point P will then be as $\sqrt{D A^2 - D P^2}$, by what was demonstrated in the first *Coroll.* compared with the twelfth *Coroll.* of the last Proposition but one. Take now $D P$ to $D A$ in the same Proportion as before those Lines were diminished; and then preserving this Proportion, restore again $D A$ and $D P$ to their former Magnitudes, and by the foregoing *Corollary*, the Velocity in P will be increased in the same Proportion as the Line $D A$ is increased: But the Quantity $\sqrt{D A^2 - D P^2}$ will also be increased in the same Proportion; for since $D P$ is as $D A$, $\overline{D P^2}$ will be as $\overline{D A^2}$, and $\overline{D A^2} - \overline{D P^2}$ as $\overline{D A^2}$, and $\sqrt{\overline{D A^2} - \overline{D P^2}}$ as $D A$: Therefore this Quantity $\sqrt{\overline{D A^2} - \overline{D P^2}}$, or its Equal $P M$, will still express the Velocity of the oscillating Body in the Point P ; whatever be the Length of the Diameter $A D C$.

COROLL. V.

The Time wherein the oscillating Body passes over any Part $A P$ of the whole Line $A D C$ will be to the Time of an entire Oscillation through that Line as the Ark $A M$ is to the Semicircle $A M C$.

For

For drawing another Perpendicular NQ infinitely near to MP , the Time wherein the infinitely small Space QP is described will be as $\frac{PQ}{PM}$, that is, as the Space PQ directly, and as the Velocity PM inversely. Join DM , and draw NR equal and parallel to PQ cutting MP in R , and the similar Triangles DMP , and MNR will give NR or PQ to MP , as MN is to MD ; therefore $\frac{PQ}{PM} = \frac{MN}{MD}$: Therefore the Time wherein the Space PQ is passed over is as the Ark MN : Therefore the Time wherein the Space AP is passed over is as the Ark AM ; and therefore this Time will be to the Time of an entire Oscillation as the Ark AM is to the Semicircle AMC .

Of the DENSITY of the AIR.

HYPOTHESIS.

“ That the Density of the Air is proportionable
“ to the compressive Force with which it is condensed.”

Though in extreme Cases this Hypothesis be attended with some Difficulties; (for it supposes the Air capable of being both condensed and rarified *in infinitum*, with some other Absurdities) yet as far as any Experiments of both kinds hitherto made can reach, it is not found liable to any exception.

LEMMA,

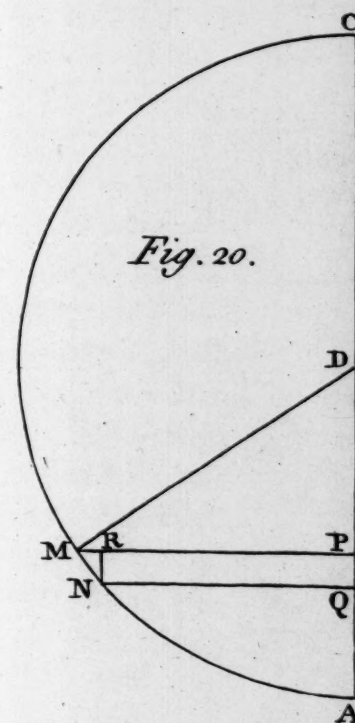
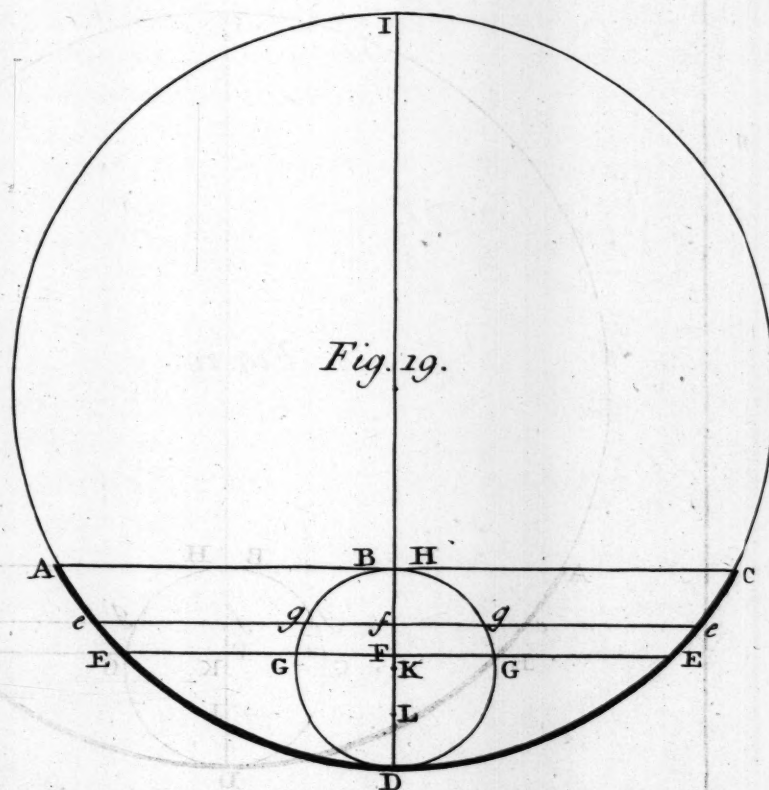
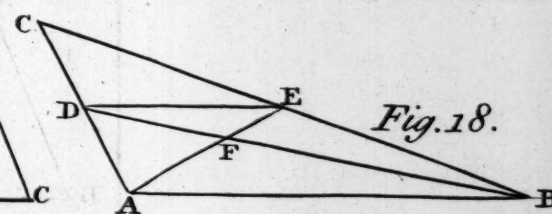
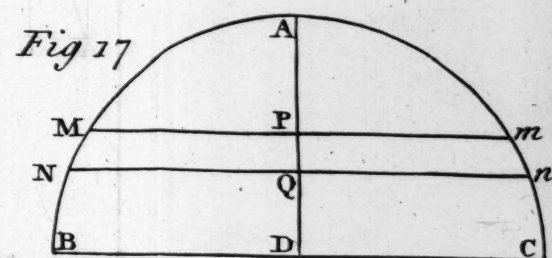
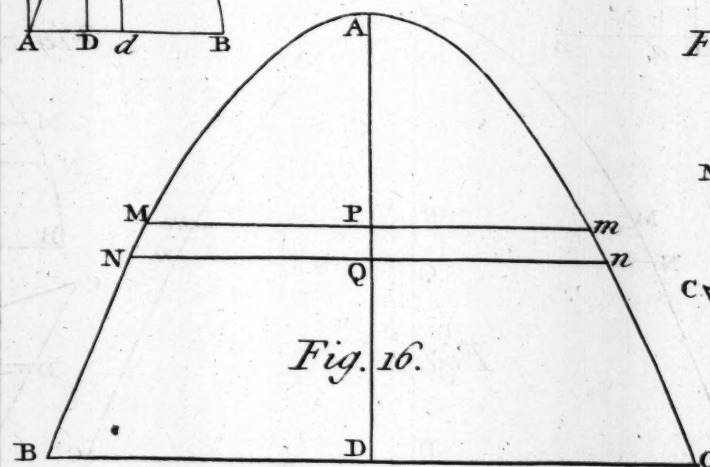
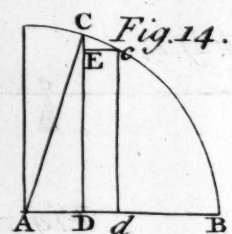
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LEMMA. Fig. 21.

Let S be the Center of the Earth, SB its Semidiameter, BC the Height of an homogeneous Atmosphere of the same Weight with our own, and every where of the same specific Gravity with that of common Air at the Surface of the Earth, and let AB be an infinitely small Part of BC : I say then, that the Density of our Air at the Point B will be to its Density at the Point A , as BC to AC .

For supposing our Atmosphere to be annihilated, let the homogeneous Atmosphere above described be placed instead of it, and let the Line BC be considered as the Axis of a Column of this Air; then it is plain, that as this Column is a homogeneous Body, the Weight incumbent upon B will be to the Weight incumbent upon A as BC to AC : But the Weight incumbent upon B from the former Atmosphere was the same as the Weight incumbent upon B from the latter, because both Atmospheres were supposed to have the same Weight. Moreover, the Weight of the Air included between A and B was the same in the former Case, as it is in the latter; because both the Space AB , and the specific Gravity of the Air included within it was supposed in both Cases to be the same. Subtract the Weight of the Air between A and B from the Weight at B , and then will remain the Weight at A in both Cases the same. Since then the Weight of Air incumbent upon B

was



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was to the Weight incumbent upon A as B C to A C, it follows, from the Hypothesis already assumed, that the Density at B is to the Density at A as B C to A C. Q. E. D.

COROLL.

Whence (*dividendo*) the Density at B will be to the Excess of the Density at B above the Density at A as B C is to B A; that is, calling B C, l ; S A, x ; and the Density at A, v ; we have v to $\dot{v}$, as l to $\dot{x}$; and therefore at the Point B, $\frac{\dot{v}}{v} = \frac{\dot{x}}{l}$.

PROBLEM XIV.

To compare the Density of the Air at any given Altitude with its Density at the Surface of the Earth, upon a Supposition that the Force of Gravity is every where the same.

Let S be the Center of the Earth as before, S B its Semidiameter, B A any given Altitude above the Surface B. Call S A, x ; the Density of the Air at A, v ; and call the Height of an homogeneous Atmosphere of the same Weight with ours, and every where of the same specific Gravity with that of common Air at the Surface of the Earth, l . Let the Altitude B A also represent Part of the Axis of a Column of the Atmosphere incumbent upon B; and supposing this Column to be resolved into an infinite Number of infinitely thin *Laminæ*, call the Thickness of that Lamina which is contiguous to the Point A, $\dot{x}$;

G

and

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and its Weight will be as $v \dot{x}$, that is, as its Density and Magnitude considered together. But the Weight of every Lamina is the Fluxion of the Pressure in the Place where it exists; and as the Density is every where proportionable to the Pressure, the Fluxion of the Density will be proportionable to the Fluxion of the Pressure, that is,

$\dot{v}$ will be every where as $v \dot{x}$; therefore $\frac{\dot{v}}{v}$ will be as $\dot{x}$, and consequently as $\frac{\dot{x}}{l}$. But at the Point

B, $\frac{\dot{v}}{v} = \frac{\dot{x}}{l}$ by the Corollary to the foregoing *Lemma*;

therefore every where else $\frac{\dot{v}}{v} = \frac{\dot{x}}{l}$: But the Fluent of $\frac{\dot{x}}{l}$ is $\frac{x}{l}$, whose Value at the Point B is $\frac{SB}{l}$, and at the Point A, $\frac{SA}{l}$, and the Difference of these

two Values is $\frac{BA}{l}$. Again, the Fluent of $\frac{\dot{v}}{v}$ is

the hyperbolic Logarithm of v (by what has been formerly demonstrated) the Value whereof at the Point B is the Logarithm of the Density at B; and at the Point A is the Logarithm of the Density at A; and the Difference of these two Values is the Logarithm of a Fraction whose Numerator is to the Denominator as the Density at B to the

Density at A; therefore $\frac{BA}{l}$ is the hyperbolic Logarithm of such a Fraction. Make 0,434294481903

$= m$, and $m \times \frac{BA}{l}$ will be *Briggs's* Logarithm of

the

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the same Fraction. Therefore whenever the Altitude BA is given, find the Fraction whose Logarithm is $m \times \frac{BA}{l}$, and the Density of the Air at B will be to the Density at A as the Numerator of that Fraction is to the Denominator. Q. E. I.

N. B. Every whole Number must here be considered as a Fraction whose Denominator is Unity.

COROLL. I.

If we call the Density of the Air at the Surface of the Earth, 1; its Density at any other Altitude BA will be expressed by a Fraction whose Logarithm is $-m \times \frac{BA}{l}$. For if $m \times \frac{BA}{l}$ be the Logarithm of the Fraction $\frac{r}{s}$, the Density at B will be to the Density at A as r to s , or as 1 to $\frac{s}{r}$, whose Logarithm is $-m \times \frac{BA}{l}$.

COROLL. II.

Whence since m and l are constant Quantities, the Logarithm of the Density of the Air at A will be as the Altitude BA .

COROLL. III.

Whence it follows, from the Nature of Logarithms, that if any Number of different Altitudes be taken in an arithmetical Progression the Den-

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sity of the Air at those Altitudes will be in a geometrical Progression; that is, it will be diminished in a continual geometrical Proportion.

COROLL. IV.

If the Density at A be given, the Altitude B A may be found thus: Let the Density at B be to the Density at A as r to s , and let t be the Logarithm of the Fraction $\frac{r}{s}$; then will $-t = -m \times \frac{BA}{l}$, and consequently B A will be equal to $l \times \frac{t}{m}$.

SCHOLIUM.

The specific Gravity of well purged Mercury is to that of Water as about $13\frac{1}{3}$ to 1; and the specific Gravity of Water is to that of Air when the Mercury in the Barometer stands at 30 Inches as 870 to 1. Put these Ratios together, and the specific Gravity of Mercury will be to that of Air, when the Mercury in the Barometer stands at 30 Inches, as 11890 to 1. Therefore the Height of an homogeneous Atmosphere of the same Weight with ours, and every where of the same specific Gravity with that of common Air at the Surface of the Earth; which in the foregoing Computation we called l , will be 356700 Inches, or 29725 Feet, or 5,63, or 5,629735 *English* Miles very nearly. Wherefore if the Altitudes l and B A be taken in Miles and Parts of a Mile, we shall have $\frac{m}{l} = 0,07714$, and $m \times \frac{BA}{l} = BA \times 0,07714$.

P R O-

PROBLEM XV.

To compare the Density of the Air at a given Altitude with its Density at the Surface of the Earth, upon a Supposition that the Force of Gravity in all Places is every where reciprocally as the Squares of their Distances from the Center of the Earth.

SOLUTION,

Using the Notation of the foregoing Solution, the Weight of an infinitely thin Lamina of Air contiguous to the Point A will now be as $\frac{v \dot{x}}{xx}$, be-

cause the Force of Gravity is as $\frac{1}{xx}$; therefore $\dot{v}$ will now be as $\frac{v \dot{x}}{xx}$, and $\frac{\dot{v}}{v}$ as $\frac{\dot{x}}{xx}$, or as $\frac{\overline{SB}^2}{l} \times \frac{\dot{x}}{xx}$;

But, at the Point B, $\frac{\dot{v}}{v} = \frac{\dot{x}}{l} = \frac{\overline{SB}^2}{l} \times \frac{\dot{x}}{xx}$; because in this Case $\frac{\overline{SB}^2}{xx} = 1$: Therefore universally $\frac{\dot{v}}{v} = \frac{\overline{SB}^2}{l} \times \frac{\dot{x}}{xx}$.

But the Fluent of $\frac{\dot{x}}{xx}$ is $\frac{-1}{x}$, whose Value at the

the Point A is $\frac{-1}{SA}$, and at the Point B, $\frac{-1}{SB}$;

and the Difference of these two Values is $\frac{AB}{SA \times SB}$:

Therefore $\frac{\overline{SB}^2}{l} \times \frac{AB}{SA \times SB}$ or $\frac{SB}{SA} \times \frac{AB}{l}$, is the hy-

perbolic Logarithm of a Fraction whose Nume-

rator is to the Denominator, as the Density at B to the Density at A: Therefore $\frac{SB}{SA} \times m \times \frac{AB}{l}$ will

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be *Briggs's* Logarithm of the same Fraction. Diminish the Altitude AB into the Altitude DB , by making as SA is to SB so AB to DB , and we shall have $\frac{SB}{SA} \times AB = DB$, and $\frac{SB}{SA} \times m \times \frac{AB}{l} = m \times \frac{DB}{l}$: Therefore if the Density at D be found by the Method of the foregoing Problem, upon a Supposition of uniform Gravity, the same will now be the Density at B upon a true Hypothesis of Gravity.

COROLL. I.

The Altitude BD may be found a little more expeditiously, by taking AS , AB , AD continual Proportionals, and then subtracting AD from AB .

COROLL. II.

If SA , SB , SE be taken continual Proportionals, BE will be equal to BD ; for since SA is to SB as SB is to SE , we shall have (*dividendo*) SA to AB as SB to BE , and alternatively SA to SB as AB to BE : But as SA is to SB , so is AB to BD , by the Construction: Therefore BE equals BD .

COROLL. III.

If several Altitudes SA be taken in musical Progression, the Density of the Air at those Altitudes will be in geometrical Progression. For Quantities are said to be in a musical Progression
I whose

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whose Reciprocals are in an arithmetical Progression; therefore if the several Altitudes SA be taken in a musical Progression, their Reciprocals SE will be in an arithmetical Progression; therefore the several Altitudes BD will be in a contrary arithmetical Progression; therefore the Densities of the Air at their respective Altitudes SA will be in a geometrical Progression.

SCHOLIUM.

To apply the Numbers of the foregoing Problem to this, the several Altitudes SB , SA , AB , BD must be taken in Miles and Parts of a Mile: In order to which observe that SB the Semidiameter of the Earth is about 3968 Miles.

Here it may not be amiss to take Notice, that if our assumed *Hypothesis*, that the Density of the Air was proportionable to the compressive Force, was strictly and universally true, our Atmosphere must be enveloped in no less than an infinite Quantity of circumambient Air to prevent its flying away; or which is the same Thing to preserve a finite Density of the Air at the Surface of the Earth. For since SA is to SB as SB to SE , if SA be infinite, that is, if SB be infinitely less than SA , SE will be infinitely less than SB , in which Case E will coincide with S , and BE or BD will be equal to BS : Therefore BD will be finite even when AB is infinite, and consequently the Density of the Air at the infinite Altitude AB will be finite, and may be determined by the Solution of this Problem.

Of the RESISTANCE of BODIES in FLUIDS.

LEMMA I.

Let s be the Space through which a heavy Body descends from Rest in a Second of Time, acquiring a Velocity that would carry it uniformly through $2s$ in the same Time; and let $2c$ be the Space described in a Second of Time by a Globe whose Diameter is d , being made to move uniformly in an uniform unelastic Fluid: Lastly, let e be an accelerating Force, which acting uniformly upon any Body will generate a Motion equal to that of the Globe, in the same Time as that wherein the Globe describes $\frac{2}{3}$ of its own Diameter: I say then that this Force e will be to the Force of the Resistance the Globe suffers from the fluid Medium, as the Density of the Globe is to the Density of the Fluid.

This is the 38th Proposition of Book II. of *Newton's Principia*, and is demonstrated by the Help of the two foregoing.

LEMMA II. A PROBLEM.

To find the Resistance of a given Globe being made to move with a given Velocity in a fluid Medium of a given Density.

Using the Notation of the foregoing Lemma, since the Globe, by its uniform Motion, describes the Space $2c$ in a Second of Time, it will describe

$\frac{2}{3}d$ in the Time $\frac{4d}{3c}$.

Let

Let a be the Weight of the Globe in *Vacuo*; then the Velocity generated in the Globe by the Weight a in a Second of Time, will be to the Velocity generated by the Force e in the same Globe, and in the same Time, as a is to e . Again, the Velocity generated in the Globe by the Force e in a Second of Time, will be to the Velocity generated in the same Globe by the same

Force e in the Time $\frac{4d}{3c}$ as one Second is to $\frac{4d}{3c}$, or as $3c$ is to $4d$. Put these two Ratios together (*viz.*) the Ratio of a to e , and the Ratio of $3c$ to $4d$, and we shall have the Velocity generated in the Globe by the Force a in a Second of Time to the Velocity generated in the same Globe by the

Force e in the Time $\frac{4d}{3c}$, as $3ac$ is to $4de$. But

the former Velocity is that whereby the Space $2s$ may be described uniformly in a Second of Time, and the latter Velocity is, by Supposition, that of the Globe whereby $2c$ is described in the same Time; therefore the former Velocity is to the latter as $2s$ to $2c$, or as s to c ; therefore as s is to c

so is $3ac$ to $4de$; therefore $e = \frac{3acc}{4ds}$; that is, the

Force e is equal to the Weight of the Globe mul-

tiplied by the Fraction $\frac{3cc}{4ds}$. But, by the forego-

ing *Lemma*, this Force is to the Resistance the Globe meets with from the Fluid as the Density of the

the Globe is to the Density of the Fluid ; or as the Weight of the Globe is to the Weight of so much of the Fluid as is equal in Bulk to the Globe ; therefore the Resistance the Globe meets with is equal to the Weight of so much of the Fluid as answers in Bulk to the Globe multiplied into the Fraction $\frac{3cc}{4ds}$; that is, if b be the relative Weight of the Globe in the Fluid, the Resistance sought will be $\overline{a-b} \times \frac{3cc}{4ds}$. Q. E. I.

COROLL.

The Resistance the Globe meets with from the Fluid will be as the Square of its Diameter, as the Square of its Velocity, and as the Density of the Fluid all together. For the Quantity $\frac{3}{4s}$ being constant, the Resistance will be as $\overline{a-b} \times \frac{cc}{d}$. Let m be the Density of the fluid Medium ; then will $a-b$, or the Weight of so much of the Fluid as is equal in Bulk to the Globe, be as md^3 ; that is, as its Density and Magnitude together : Therefore the Resistance will be as $md^3 \times \frac{c^2}{d}$, that is, as $md^2 c^2$.

LEMMA.

LEMMA III. A PROBLEM.

To find the greatest Velocity a given Globe can acquire in descending by its relative Weight in a Fluid of a given Density.

This is nothing else but to find what Velocity a Globe ought to move with to meet with a Resistance equal to its own relative Weight in the Fluid : For it is certain, that wherever this happens, the Globe must be incapable of all further Acceleration. Therefore supposing all Things as in the two foregoing *Lemmata*, let $2c$ be the Space described in a Second of Time by the Velocity sought, and the Resistance will be $\overline{a-b} \times \frac{3cc}{4ds}$, as above. Make this Resistance equal to b the relative Weight of the Body, and we shall have $4cc = \frac{b}{a-b} \times \frac{4}{3} ds$. Let m be a mean Proportional between $\frac{4}{3} d$, and the Space s , and we shall have $4cc = 4mm \times \frac{b}{a-b}$, and $2c = 2m \times \sqrt{\frac{b}{a-b}}$: Therefore the greatest Velocity the Globe can acquire will be such as will carry it uniformly through the Space $2m \times \sqrt{\frac{b}{a-b}}$ in a Second of Time. Q. E. I.

COROLL.

COROLL.

In the same Fluid, the Square of the greatest Velocity will be as $\frac{b}{d}$; For in this Case $a-b$ will be as d^3 ; and therefore the Quantity $\frac{b}{a-b} \times \frac{1}{3} d s$ will be as $\frac{b}{d}$.

LEMMA IV. A PROBLEM.

To find how long, and how far, a given Globe ought to descend by its comparative Weight in a Medium of a given Density but without Resistance, to acquire the greatest Velocity it is capable of in descending with the same Weight, and in the same Medium, with Resistance.

Let f be the Space and g the Time required; then will the Square of the Velocity acquired in falling through the Space s by the Weight a be to the Square of the Velocity acquired in falling through the Space f by the Weight b as $a s$ is to $b f$. (by the Theory of the Descent of Bodies.) But the Square of the former Velocity is to the Square of the latter as $4 s s$ is to $\frac{b}{a-b} \times \frac{1}{3} d s$ by the third Lemma: Therefore $a s$ is to $b f$, as $4 s s$ is to $\frac{b}{a-b} \times \frac{1}{3} d s$; whence $f = \frac{a}{a-b} \times \frac{4}{3} d$; that is, the Space f is to $\frac{4}{3}$ of the Diameter of the Globe as the Density of the Globe is to the Density of the Fluid. Q. E. I.

Again,

Again, since the Space through which a Body is uniformly accelerated is as the accelerating Force multiplied into the Square of the Time of its Continuation, we shall have the Space s to the Space f , as $a \times 1$ (that is, as a multiplied into the Square of the Time of one Second) to $b \times g g$; whence $g g$ will be found equal to $\frac{a}{b} \times \frac{f}{s}$; and g equal to $\sqrt{\frac{af}{bs}}$; that is, the Time sought will be to the Time of one Second, in a subduplicate Ratio of af to bs . Q.E.I.

COROLL. I.

If a given Ball was to move in a strait Line through a fluid Medium of a given Density, and if its first Resistance was to continue all along the same till it had quite destroyed the Motion of the Ball, the Space through which the Ball would pass to suffer this Effect would be always the same: For it would always be as the Square of the first Velocity directly, and as the Resistance, that is, as the Square of the same Velocity inversely. Therefore if this Space be known in any one particular Case, it will be known in all others. But in the Case where the Square of the first Velocity is $\frac{b}{a-b} \times \frac{1}{3} ds$, this Space is known to be the Space f ; because a Continuation of the first Resistance in this Case will have the same Effect as a like Continuation of the relative Weight b ; therefore in all other Cases the Space abovementioned will be f .

COROLL.

COROLL. II.

The Time wherein the abovementioned Effect will be produced will be that wherein the Space $2f$ will be uniformly described by the first Velocity.

LEMMA V.

The Fluent of a Fraction whose Numerator is the Fluxion of the Denominator, is the *hyperbolic Logarithm* of the Denominator: For it has been proved already that the Fluent of $\frac{\dot{z}}{z}$ is the hyperbolic Logarithm of z .

COROLL. I.

Therefore the Fluent of this Fluxion $\frac{2v\dot{v}}{mm-vv}$ is the negative hyperbolic Logarithm of $mm-vv$, that is, it is the hyperbolic Logarithm of $\frac{1}{mm-vv}$.

COROLL. II.

And the Fluent of this Fluxion $\frac{2m\dot{v}}{mm-vv}$ is the hyperbolic Logarithm of $\frac{m+v}{m-v}$. For $\frac{2m\dot{v}}{mm-vv} = \frac{\dot{v}}{m+v} + \frac{\dot{v}}{m-v}$; and therefore the Fluent will be the hyperbolic Logarithm of $m+v$, minus the hyperbolic Logarithm of $m-v$; that is, it will be the hyperbolic Logarithm of the Fraction $\frac{m+v}{m-v}$.

P R O-

PROBLEM XVI.

Let a given Ball passing in a strait Line through a fluid Medium of a given Density, be resisted every where in Proportion to the Square of its Velocity: *It is required, having given the Velocity of the Ball at its first setting out, to find not only the Motion it will lose, but also the Space it will pass through in a given Time afterwards.*

Let d be the Diameter of the Globe, and let the Space f be taken in Proportion to $\frac{4}{3}d$, as the Density of the Globe is to the Density of the Fluid; let M be the first Velocity, and R the first Resistance of the Ball, and let T be the Time wherein the Space $2f$ may be uniformly described by the first Velocity M ; then will T be also the Time wherein the first Resistance R , if uniformly continued, would entirely destroy the first Motion of the Ball; by the Corollaries to the fourth *Lemma*. Let s be the Space described by the Ball in any indeterminate Time t , from its first setting out, and let v be the Velocity and r the Resistance at the End of the Time t : Then since the Velocity generated or destroyed by any uniform accelerating or retarding Force is as the Force multiplied into the Time of its Action, we shall have the Velocity M to the Velocity v as $R \times T$ the Product of the Time and Force whereby the former Velocity may be destroyed, to $r \times t$ the Product of the Time and Force whereby the latter Velocity is destroyed. But R is to r as $M M$ is to $v v$ by the *Hypothesis*; there-

therefore RT is to r ; as MMT is to vv ; therefore M is to $\dot{v}$ as MMT is to vv ; therefore $i = MT \times \frac{\dot{v}}{vv}$. But the Fluent of i is t , and the Fluent of $MT \times \frac{\dot{v}}{vv}$ is $MT \times \frac{-1}{v}$, whose Value at the Beginning of the Time t is $MT \times \frac{-1}{M}$; and at the End of the Time t is $MT \times \frac{-1}{v}$; and the Difference is $MT \times \frac{M-v}{Mv} = \frac{TM-Tv}{v}$; therefore $t = \frac{TM-Tv}{v}$; and consequently v , or the Velocity at the End of the Time will be $\frac{TM}{T+t}$; and therefore the Velocity destroyed by the Resistance in the Time t will be such a Part of the whole M as is expressed by the Fraction $\frac{tM}{T+t}$. Q. E. I.

Now to find the Space s we have the following Proportion (*viz.*) As the Space $2f$ is the Space i , so is MT the Product of the Time and Velocity whereby the former Space may be uniformly described to v ; the Product of the Time and Velocity whereby the latter Space is described; therefore $i = \frac{2f}{MT} \times vv$. Instead of i substitute its Value $MT \times \frac{\dot{v}}{vv}$, and we shall have $i = 2f \times \frac{\dot{v}}{v}$. But the the Fluent of i is s , and the Fluent of $\frac{\dot{v}}{v}$ is the hyperbolic Logarithm of v , which at the Beginning of

of the Time t is the hyperbolic Logarithm of M , and at the End of the Time t is the hyperbolic Logarithm of v ; and the Difference is the hyperbolic Logarithm of $\frac{M}{v}$; therefore s is the hyper-

bolic Logarithm of $\frac{M}{v}$ multiplied into $2f$. Instead of v substitute its Value $\frac{TM}{T+t}$ as above found, and we shall have s equal to the hyperbolic Logarithm of $\frac{T+t}{T} \times 2f$. Now as the Ball in the

Time T by its first Velocity M would describe uniformly the Space $2f$, it would in the Time t

by the same Velocity M describe the Space $2f \times \frac{t}{T}$;

therefore the Space s actually described by the Ball in the Time t , with Resistance, will be to the Space it would have described in the same Time, had it moved all along with its first Velocity, without any

Resistance, as the hyperbolic Logarithm of $\frac{T+t}{T}$,

is to the Fraction $\frac{t}{T}$. Make $m = 0,434294481903$,

and we shall have the hyperbolic Logarithm of

$\frac{T+t}{T}$ equal to Briggs's Logarithm of the same

multiplied into $\frac{1}{m}$, that is, into $2,302585093$;

therefore the Space described in the Time t with Resistance is to the Space that would have been described in the same Time without Resistance, as the

H common

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common Logarithm of $\frac{T+t}{T}$ multiplied into 2,302585093 is to the Fraction $\frac{t}{T}$. Q. E. I.
See *Newton's Principia*, Edit. 3. Pag. 355.

SCHOLIUM.

In the foregoing Solution the Space s was found equal to the hyperbolic Logarithm of $\frac{T+t}{T}$ multiplied into $2f$, which is equal to *Briggs's* Logarithm of the same Fraction multiplied into $\frac{2f}{m}$. Let $t=T$, and let the Density of the Fluid the Ball moves in be the same with that of the Ball; then will the Logarithm of $\frac{T+t}{T}$ be the Logarithm 2,30103; and the Space $2f$ will be equal to $\frac{2}{3}d$, and $\frac{2f}{m} = 6,140227d$; therefore the Logarithm of $\frac{T+t}{T} \times \frac{2f}{m}$, or the Space s will be 1,8484 d , which is somewhat less than $2d$. Again, The Motion lost which was $\frac{tM}{T+t}$ will in this Case be $\frac{1}{2}M$; therefore a Ball projected in a Fluid of the same Density with itself will lose half its Motion, before it has passed through a Space equal to two of its own Diameters.

PROBLEM XVII.

Let a given Ball be supposed to fall from Rest in a Fluid of a given Density, being every where accelerated by the Excess of its comparative Weight above the Resistance of the Fluid: *It is required to assign the Space through which it will fall, and the Velocity it will acquire in any given Time.*

Let s be the Descent made, and v the Velocity acquired in any indeterminate Time p , and let Unity, or 1, represent both the relative Weight of the Ball, and the greatest Velocity it can acquire in descending through this Fluid: Therefore 1 will also be the Resistance of the Fluid answering to the Velocity 1, and vv the Resistance answering to the Velocity v , and $1-vv$ will be the unequal Force whereby the Motion of the Ball will be constantly accelerated. Let f be the Space, and g the Time of the Descent necessary to be made for the same Body to acquire by its relative Weight the same Velocity 1 in the same Medium without Resistance: Then, since the Velocity 1 is generated in the Time g by the Force 1, and since the Velocity v is generated in the Time p by the Force $1-vv$, we shall have 1 to v as $1 \times g$ is to $1-vv \times p$; whence $\frac{2p}{g} = \frac{2v}{1-vv}$: But the Fluent of $\frac{2p}{g}$ is $\frac{2p}{g}$, and the Fluent of $\frac{2v}{1-vv}$ is the hyperbolic Logarithm of the Fraction $\frac{1+v}{1-v}$; (by the second Corollary to

the fifth *Lemma*) which, at the Beginning of the Fall, is nothing; therefore $\frac{2p}{g}$ is the hyperbolic Logarithm of $\frac{1+v}{1-v}$, and if (as before) m be put equal to .4342944819, then $\frac{2pm}{g}$ will be the common Logarithm of $\frac{1+v}{1-v}$. Let n be the natural Number answering to the Logarithm $\frac{2pm}{g}$, and we shall have $\frac{1+v}{1-v} = n$; whence v will be equal to $\frac{n-1}{n+1}$; that is, the Velocity v acquired in the Time p will be such a Part of the greatest Velocity 1 as is expressed by the Fraction $\frac{n-1}{n+1}$; or (which is the same thing) the Velocity v will be to the greatest Velocity as $n-1$ is to $n+1$. Q. E. I.

Again, since the Space $2f$ can be described by the Velocity 1 in the Time g , and since the Space; is described by the Velocity v in the Time p , we shall have $2f$ to; as $1g$ is to vp ; whence $=fv \times \frac{2p}{g}$. Instead of $\frac{2p}{g}$ put its Value above found (viz.) $\frac{2v}{1-vv}$, and we shall have $=f \times \frac{2vv}{1-vv}$.

But the Fluent of v is s , and the Fluent of $\frac{2vv}{1-vv}$ is the hyperbolic Logarithm of $\frac{1}{1-vv}$, by the first *Corollary* to the fifth *Lemma*; therefore the Space s is the hyperbolic Logarithm of $\frac{1}{1-vv}$ multiplied into

into f ; or it is *Briggs's* Logarithm of the same Fraction multiplied in $\frac{f}{m}$. But v has already been proved equal to $\frac{n-1}{n+1}$; therefore $\frac{1}{1-vv}$ is equal to $\frac{nn+2n+1}{4n}$; therefore now we may say that the Space s is the Logarithm of $\frac{nn+2n+1}{4n}$ multiplied into $\frac{f}{m}$. But this Fraction $\frac{nn+2n+1}{4n}$ is the Product of two others; whereof one is $\frac{n+1}{n}$, and the other is $\frac{n}{n+1}$. Let l be the Logarithm of $\frac{n+1}{n}$; then will $2l$ be the Logarithm of $\frac{n+1}{n}$. Let k be the Logarithm of 4 ; then since the Logarithm of n was before found equal to $\frac{2pm}{g}$, the Logarithm of $\frac{n}{4}$ will be $\frac{2pm}{g} - k$. Put these two Logarithms together, and we shall have the Logarithm of the Fraction $\frac{nn+2n+1}{4n} = \frac{2pm}{g} - k + 2l$; therefore the Space $s = \frac{2pm}{g} - k + 2l \times \frac{f}{m} = \frac{2pf}{g} - \frac{k}{m} \times f + \frac{2l}{m} \times f = \frac{2pf}{g} - 1,3862943611f + 4,605170186/f$.

Q. E. I.

See *Newton's Principia*, L. ii. Prop. 40.

SCHOLIUM,

By several Experiments made upon the Descent of Bodies both in Water and Air, this Theory of Resistances is not only sufficiently confirmed, but it is also further demonstrated that, in these Fluids at least, the Resistance that might arise from other Causes is either none at all, or very inconsiderable in Comparison of that here considered which arises from the *Vis inertiae* of the several Particles whereof all Bodies, as well Fluids as others, are composed. See the *Scholium* to the fortieth *Prop.* of the second Book of the *Principia*.

Of the ACTION of a prolate SPHEROID upon a Particle of Matter placed any where in its Axis produced; necessary for understanding the Figure of the Earth.

PROP. I. A THEOREM. Fig. 22.

Let $EF (=x)$ be a Line given in Position, and let $FG (=y)$ be supposed to move always perpendicular to EF , and let $a - 2bx - cxx = cyy$, the Quantities a , b , and c being constant: I say then that the Point G will describe the Circumference of a Circle whose Center is somewhere in the Line EF , or in the Line EF produced.

DEMONSTRATION.

Produce the Line FE out from E to D, so that ED may be equal to $\frac{b}{c}$, and call DF, z ; then will EF or $x = z - \frac{b}{c}$, and $-2bx = -2bz + \frac{2bb}{c}$, and $-cx^2 = -cz^2 + 2bz - \frac{bb}{c}$, and $a - 2bx - cx^2$ or $yy = a - 2bz + \frac{2bb}{c} - cz^2 + 2bz - \frac{bb}{c} = a + \frac{bb}{c} - cz^2 = \frac{bb+ac}{c} - cz^2$; therefore $yy = \frac{bb+ac}{cc} - zz$. Make $\frac{bb+ac}{cc} = rr$, and we shall have $yy = rr - zz$; which is the Nature of a Circle whose Center is D and Semidiameter r . Q. E. D.

Note. If the Coefficient of x is $+2b$, the Point D is to be taken on the other Side of E.

PROP. II. A THEOREM. Fig. 23.

Let $ABab$ be an *Ellipse*, Aa the greater Axis, Bb the lesser, C the Center, F and f the Foci, P a Point any where in the lesser Axis Bb produced from B , PD a Line meeting the Ellipse any where in the Point D , EDG a Line drawn through D perpendicular to the Axis Bb in E ; and let the Length EG be so taken that the Line PD may always have the same Proportion to EG , that FC hath to CB : I say then, *That the Locus of the Point G will be in the Circumference of a Circle whose Center is somewhere in the Line Pb, or in that Line produced from b.*

H 4

DE.

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DEMONSTRATION.

Call CA, t ; CB, c ; CF, m ; CP, d ; CE, x ; and we shall have $PE = d +$ or $-x$, and $PE^2 = dd +$ or $-2dx + xx$. Moreover it is evident, from the Nature of the *Ellipse*, that $ED^2 = tt - \frac{ttxx}{cc}$; wherefore PD^2 or $PE^2 + ED^2 = dd + tt +$ or $-2dx + \frac{ttxx}{cc}$. But $xx = \frac{ccxx - ttxx}{cc}$; $\therefore$ PD^2 or $EG^2 = dd + \frac{m^2 m^2}{cc}$; therefore PD^2 or $EG^2 = dd +$ or $-2dx + \frac{m^2 m^2}{cc}$. Therefore by the foregoing Proposition, the Locus of the Point G is the Circumference of a Circle whose Center is in the Line Pb, or in that Line produced, Q.E.D.

COROLL. I.

To construct the Locus of the Point G; let BK, CI, and bk be Perpendiculars to the Axis Bb. Let BK be taken equal to $PB \times \frac{c}{m}$, and bk equal to $Pb \times \frac{c}{m}$; then it is evident that the Points K and k will be in the Circumference of the Circle required. And since BK is to bk as PB is to Pb, it is also further evident, that the Line PK that passes through the Points P and K will also pass through k. Let the Line Pk cut the Perpendicular CI in I, and draw IH perpendicular to PI cutting

cutting the Line Pb (produced if need be) in H , then will a Circle upon the Center H , and at the Distance HK or Hk be the Circle required.

COROLL. II.

The Triangles PBK , Pbk , PCI , PIH , FCB are all similar to one another; for they have all one right Angle, and the Sides about it proportionable. PB is to BK as FC to CB by the Construction.

COROLL. III.

Join PF and HK , and the Triangle HIK will be similar to the Triangle PCF ; for HI is to IP , as BC is to CF ; and IP is to IK as PC is to BC ; wherefore, *ex æquo perturbatè*, HI will be to IK as PC to CF .

PROP. III. A PROBLEM.

Let the Ellipse $ABab$ by a Revolution about its lesser Axis Bb generate a prolate Spheroid whose equal Particles attract at all Distances with Forces reciprocally as the Squares of those Distances: *It is required to determine the Law of the Force wherewith a Particle of Matter, placed any where in the Axis Bb produced, will be attracted by the whole Spheroid.*

SOLUTION.

Retaining the Notation and Construction of the foregoing Proposition; From the Point e in the
Line

Line CEe let a new Ordinate edg be drawn infinitely near the former EDG , and let the Spheroid be imagined to be resolved into an infinite Number of infinitely thin circular *Laminae*, whose Planes are all perpendicular to the Axis Bb , and let the Space $EDde$ by its Revolution about Ee generate one of these *Laminae*: Then it is evident from the first *Corollary* of the ninetieth Proposition of the first Book of the *Principia*, that the Attraction of this *Lamina* will be as $Ee - \frac{PE \times Ee}{PD}$: But $PE = PH - HE$; therefore the Attraction of the *Lamina* will be as $Ee + \frac{HE \times Ee}{PD} - \frac{PH \times Ee}{PD}$: Therefore the Fluent of this Fluxion, which is to represent the Attraction of the whole Spheroid, will consist of three Parts, which must be found separately thus:

First, The Sum of all the Fluxions Ee , at least so much of that Sum as makes for our Purpose, is Bb or $2c$.

Secondly, In order to find the Fluent of $\frac{HE \times Ee}{PD}$;

draw gR equal and parallel to Ee , and cutting EG in R , and join GH ; Then will the Triangles GRg , and HEG be similar, and we shall have Rg , or Ee , to GR as EG is to HE : whence

$$HE \times Ee = EG \times GR = PD \times GR \times \frac{c}{m}; \text{ therefore } \frac{HE \times Ee}{PD} = GR \times \frac{c}{m}; \text{ and the Sum of all the}$$

Fluxions

Fluxions $\frac{H E \times E e}{P D}$ will be equal to the Sum of all the Fluxions $G R \times \frac{c}{m}$ generated in the same Time.

But as much of this latter Sum as makes for our Purpose is $\overline{b k} - \overline{B K} \times \frac{c}{m} = \overline{P b} - \overline{P B} \times \frac{c c}{m m} = \overline{B b} \times \frac{c c}{m m}$.

Therefore the second Part of our Fluent is

$\overline{B b} \times \frac{c c}{m m}$. Add this to $\overline{B b}$ the first Part already found, and they will make $\overline{B b} \times 1 + \frac{c c}{m m} = \overline{B b} \times \frac{m m + c c}{m m} = \overline{B b} \times \frac{t t}{m m} = \frac{2 c t^2}{m m}$. Call $c t^2$, which is

always proportionable to the Magnitude of the Spheroid, q ; and then the two first Parts of the

Fluent added together will make $\frac{2 q}{m m}$.

Thirdly, We come now to the third Part of the Fluent, which is to be subtracted from the former,

and which is the Fluent of $\frac{P H \times E e}{P D}$; and this

also will be obtained by the Help of the former similar Triangles $G R g$ and $H E G$; where $R g$, or $E e$, is to $G g$, as $E G$ is to $G H$; therefore

$G H \times E e = E G \times G g = P D \times G g \times \frac{c}{m}$; therefore

$\frac{E e}{P D} = \frac{G g}{G H} \times \frac{c}{m}$; and $\frac{P H \times E e}{P D} = P H \times \frac{c}{m} \times \frac{G g}{G H}$;

and the Fluent of the former is equal to the Fluent

of the latter, which is $P H \times \frac{c}{m} \times \frac{K G k}{G H}$.

Join

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Join Pf and Hk , and upon the Center P , and with the Radius PC , sweep the Ark LCl , cutting PF and Pf in L and l respectively; and because of the Angle $LP l$ equal to the Angle $KH k$, (by the third *Coroll.* to the last Prop.) the Arks Kgk and LCl will be similar, and we shall have $\frac{KGk}{GH} = \frac{LCl}{PC}$, and $PH \times \frac{c}{m} \times \frac{KGk}{GH} = \frac{PC}{m} \times \frac{PH}{m} \times LCl$. But PH is to PI , as t to m ; therefore PH is to PC , as tt to mm , and $\frac{PH}{PC} = \frac{tt}{mm}$; therefore $\frac{PH}{PC} \times \frac{c}{m} \times LCl = \frac{ctt}{m^3} \times LCl = \frac{2q}{m^3} \times CL$.

In the Tangent CF , let a Line as CO be taken equal to the Ark CL , and the third Member of the Fluent to be subtracted from the two former will be $\frac{2q}{m^3} \times CO$. Subtract therefore this Fluent $\frac{2q}{m^3} \times CO$ from the Sum of the two former, which was $\frac{2q}{mm}$ or $\frac{2q}{m^3} \times CF$, and there will remain $\frac{2q}{m^3} \times FO$ for the Attraction of the whole Spheroid: Therefore the Attraction of the Spheroid will be as its Magnitude, and the Line FO directly, and as the Cube of the Line CF inversely.

This is upon a Supposition that the Matter of the Spheroid is uniform, and that its Density continues always the same: But if we suppose the Density to vary, then the Attraction of the Spheroid will be as its Magnitude and Density and the
Line

Line FO directly, and as the Cube of the Line CF inversely; that is, the Attraction will be as the Quantity of Matter in the Spheroid, and the Quantity $\frac{FO}{CF^3}$ together. Q. E. I.

COROLL. I.

Since CO is equal to an Ark whose Tangent is CF ; by a Series formerly computed it follows, that CO will be equal to $CF - \frac{CF^3}{3CP^2} + \frac{CF^5}{5CP^4}$

$-\mathcal{E}c$. Therefore $FO = \frac{CF^3}{3CP^2} - \frac{CF^5}{5CP^4} \mathcal{E}c$.

and $\frac{FO}{CF} = \frac{1}{3CP^2} - \frac{CF^2}{5CP^4} \mathcal{E}c$. Let CF vanish, and then the Term $\frac{CF^2}{5CP^4}$ and all that follow it, represented by $\mathcal{E}c$. will vanish with it, and we shall have $\frac{FO}{CF^3} = \frac{1}{3CP^2}$.

COROLL. II.

The Attraction of the Spheroid will be to the Attraction of the same Matter condensed into an infinitely small Particle in the Center, as $\frac{FO}{CF^3}$ is to $\frac{1}{3CP^2}$. This follows from the foregoing Corollary; for if the Figure of the condensed Particle

was

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was a Spheroid, CF would be evanescent in respect of PC, which is supposed to continue the same: But the Attraction of the Particle will be the same in this Case, whatever Figure it puts on.

COROLL. III.

The Attraction of every Sphere whose Matter is uniform is as the Quantity of Matter it contains directly, and as the Square of the Distance of the attracted Particle from the Center of the Sphere inversely.

COROLL. IV.

Every Sphere attracts with the same Force as if all its Matter was accumulated into its Center: For, by the foregoing Corollary, the Attraction of a Sphere depends only upon the Distance of the attracted Particle from the Center of the Sphere, and upon the Quantity of Matter in the Sphere, and not at all upon its Extent.

COROLL. V.

Since the Quantity of Matter in the Spheroid is to the Quantity of Matter of a homogenous Sphere upon the same Axis Bb as $BC \times \overline{AC}^2$ is to $\overline{BC}^3$, or as $\overline{AC}^2$ is to $\overline{BC}^2$; it follows that the Attraction of the Spheroid will be to the Attraction of a homogenous Sphere upon the same Axis as $\overline{AC}^2 \times \frac{FO}{CF}$ to $\frac{BC^2}{3CP^2}$.

SCHOLIUM I.

If the two *Axes* of the generating Ellipse be not very unequal, and the Proportion of the last *Corollary* is to be computed, there is no Necessity of having Recourse to trigonometrical Tables: For the Computation may be performed much more expeditiously by the Series of the first *Coroll.*

thus: Make $\overline{AC^2} = p$, $\frac{p \times \overline{CF^2}}{\overline{CP^2}} = q$, $\frac{q \times \overline{CF^2}}{\overline{CP^2}} = r$, $\frac{r \times \overline{CF^2}}{\overline{CP^2}} = s$ &c. Then will the Attraction of the

Spheroid be to the Attraction of a homogeneous

Sphere upon the same *Axis*, as $\frac{p}{3} - \frac{q}{5} + \frac{r}{7} - \frac{s}{9}$ &c. is to $\frac{\overline{BC^2}}{3}$. *Ex. Gr.* Let BC be 100, and AC 101,

and consequently $\overline{CF^2}$ 201, and let the Particle P be placed at the Vertex B ; then we shall have $p = 10201$, $q = 205$, and $r = 4$, and $\overline{BC^2} = 10000$, and the Attraction of the Spheroid will be to the

Attraction of the Sphere as $\frac{10201}{3} - \frac{205}{5} + \frac{4}{7}$ is to $\frac{10000}{3}$; that is, as $10201 - 123 + 2$ is to 10000, that is, as 10080 is to 10000, or 126 to 125.

SCHOLIUM II. Fig. 24.

Let the attracted Particle P be placed within the Spheroid, suppose between B and C , and let BK and bk be taken as before both on the same Side
the

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the Axis, and the Lines PK and Pk will now be different Lines. Moreover in the Segment whose

Axis is Pb , the Fluxion $Ee - \frac{PE \times Ee}{PB}$ will still

be equal to $Ee + \frac{HE \times Ee}{PD} - \frac{PH \times Ee}{PD}$ as before.

But in the other Segment whose Axis is PB , the Fluxion is $Ee - \frac{HE \times Ee}{PD} + \frac{PH \times Ee}{PD}$: Therefore

drawing PT perpendicular to Bb as far as the Circumference KIk , the Attraction of the Segment

whose Axis is Pb will be $Pb - \overline{bk} - \overline{PT} \times \frac{c}{m} - PH \times \frac{c}{m} \times \frac{Tk}{HK}$; and the Attraction of the other Seg-

ment whose Axis is PB will be $PB - \overline{BK} - \overline{PT} \times \frac{c}{m} + PH \times \frac{c}{m} \times \frac{TK}{HK}$. Subtract the Attraction of

this latter Segment from that of the former, that is, subtract PB from Pb , and there will remain $2PC$.

Again, subtract $\overline{BK} - \overline{PT} \times \frac{c}{m}$ from $\overline{bk} - \overline{PT} \times \frac{c}{m}$,

and there will remain $\overline{bk} - \overline{BK} \times \frac{c}{m} = 2PC \times \frac{cc}{mm}$.

Lastly, subtract $PH \times \frac{c}{m} \times \frac{TK}{HK}$ from $-PH \times \frac{c}{m} \times \frac{Tk}{HK}$,

and there will remain $-PH \times \frac{c}{m} \times \frac{KTk}{HK}$.

Now that this last Part is as the Distance PC as well as the two former, I thus demonstrate: Produce the Chord Kk , and the Axis Bb till they meet

meet in Q , and HI will be to IQ , as CI to CQ ; and IQ will be to IK , as CQ to CB ; therefore, *ex aequo*, HI will be to IK , as CI to CB ; that

is, in a constant Ratio; for CI or $\frac{BK + bk}{2}$ or

$\frac{1}{2} Bb \times \frac{c}{m}$ is a constant Quantity; therefore the

Triangle HIK , and consequently also the Triangle HKk will be given in *Specie*: Therefore the Angle KHI will be the same, wherever the Particle P is placed between B and b : Therefore

$\frac{KTk}{HK}$ will be a constant Quantity; Therefore

$PH \times \frac{c}{m} \times \frac{KTk}{HK}$, or $2PC \times \frac{tt}{mm} \times \frac{c}{m} \times \frac{KTk}{HK}$ will be as PC .

Of SECOND, THIRD, and FOURTH FLUXIONS and FLUENTS.

Let $a b$ represent any finite Quantity of Time terminated by the two Instants a and b , and let x be the Value of any variable Quantity at the Instant a , whose Velocity at that Instant is supposed to be such, that if the Quantity was to flow uniformly with that Velocity, during the whole Time ab , it would gain in that Time $\dot{x}$: Then will $\dot{x}$ be the Fluxion of the Quantity x , according to the Idea of Fluxions formerly given; and every other Quantity depending upon x will have a different Velocity at the Instant a . Thus if y be made equal

I

to

to x^m , the Velocity of y at the Instant a will be such as in the Time ab would gain uniformly $m\dot{x} \times x^{m-1}$; and therefore $m\dot{x} \times x^{m-1}$ is called the *first Fluxion* of y .

Again, the Velocity of this Quantity $m\dot{x} \times x^{m-1}$ at the Instant a is such, as in the Time ab would gain uniformly $m \times \overline{m-1} \times \dot{x}^2 \times x^{m-2}$; and therefore this is called the *second Fluxion* of y .

Thus again, the Velocity of this Quantity $m \times \overline{m-1} \times \dot{x}^2 \times x^{m-2}$ at the Instant a is such, as would gain uniformly in the Time ab the Quantity $m \times \overline{m-1} \times \overline{m-2} \times \dot{x}^3 \times x^{m-3}$; and therefore this Quantity is called the *third Fluxion* of y .

And for the same Reason, the *fourth Fluxion* of y will be $m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3} \times \dot{x}^4 \times x^{m-4}$; and so on *ad infinitum*; except when m is a whole Number and Affirmative, in which Case the Series of Fluxions will at last break off.

On the other hand, let us in the next Place inquire, what Quantity will have such a Velocity at the Instant a , as that, if it was to flow uniformly with that Velocity during the whole Time ab , it would gain y or x^m . Now that Quantity

we shall find to be $\frac{1}{x \times \overline{m+1}} \times x^{m+1}$; according to what has formerly been shewn concerning Fluxions; and therefore $\frac{1}{x \times \overline{m+1}} \times x^{m+1}$ is called the *first Fluent* of y .

Again,

Plate IV.

Fig. 21.

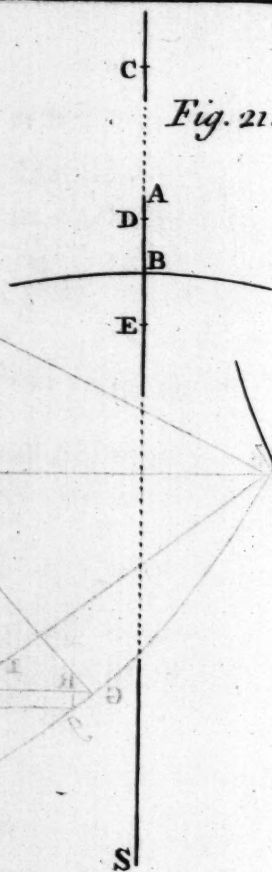


Fig. 23.

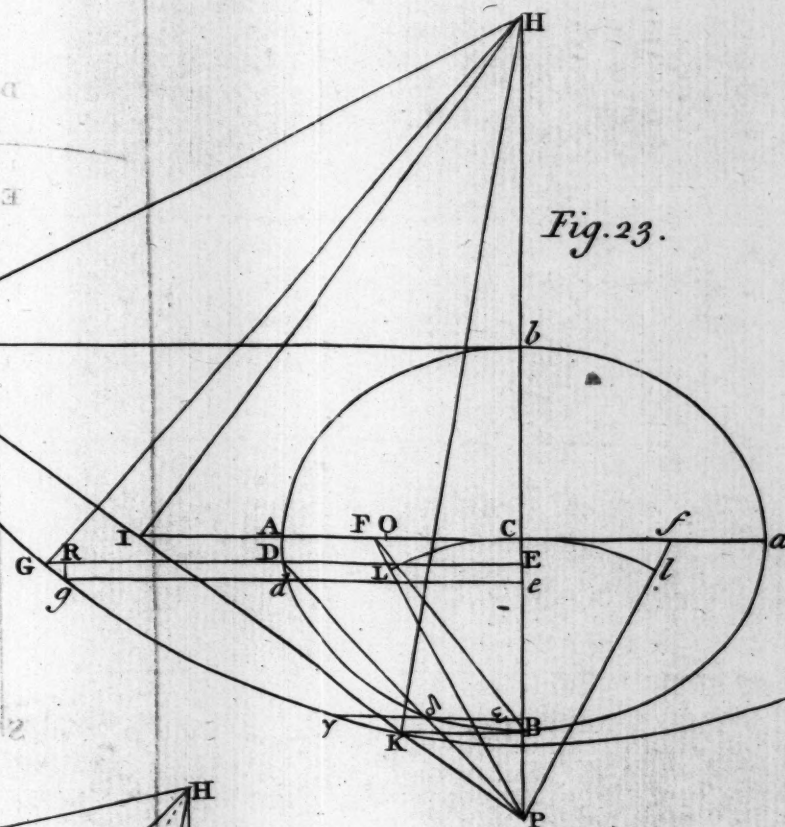


Fig. 24.

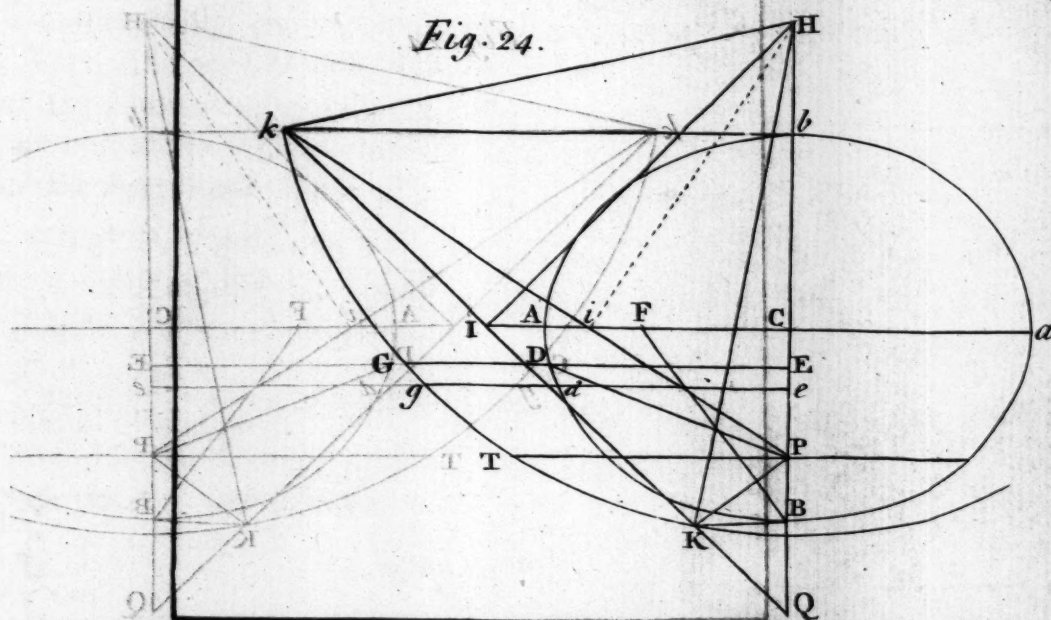


Fig. 22.



Again, if it be inquired what Quantity at the Instant a has such a Velocity as in the Time $a b$

would gain uniformly $\frac{1}{x \times m+1} \times x^{m+1}$, it will be

found to be $\frac{1}{x^2 \times m+1 \times m+2} \times x^{m+2}$; and there-

fore this Quantity is called the *second Fluent* of y .

In like Manner, the *third Fluent* of y will be

$\frac{1}{x^3 \times m+1 \times m+2 \times m+3} \times x^{m+3}$, and so on *ad infinitum*; unless m is whole Number and Negative; in which Case the Series of Fluents will also break off.

SCHOLIUM.

The radical Quantity of all these Powers, which in the present Case is x , is generally supposed to flow uniformly, and consequently to have no Fluxions beyond the first: And in this Case it is usual to put Unity for the Fluxion of x , that its several Powers may not come into the Computation. If this be done, the *first, second, third, &c.* Fluxions of y will be $m \times x^{m-1}$, $m \times m-1 \times x^{m-2}$, $m \times m-1 \times m-2 \times x^{m-3}$, &c. respectively: And the *first, second, third, &c.* Fluents of y will

be $\frac{1}{m+1} \times x^{m+1}$, $\frac{1}{m+1 \times m+2} \times x^{m+2}$,

$\frac{1}{m+1 \times m+2 \times m+3} \times x^{m+3}$, &c. respectively.

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This Doctrine thoroughly understood will be very useful in extracting the Root of a fluxional Equation, as will hereafter be shewn.

The Method of extracting the Roots of fluxional Equations, by comparing the analogous Terms of Series, illustrated by Examples.

EXAMPLE I.

In an equilateral *Hyperbola* whose *Power* is Unity, let z be the asymptotic Area comprehended between two Ordinates whose Abscissae are 1 and $1+x$, and we shall have this fluxional Equation

(viz.) $\dot{z} = \frac{\dot{x}}{1+x}$. Let it now be required to extract the Root z of that fluxional Equation ; that

is, let it be required to express the Value of z by a Series consisting of the several Powers of x .

Here because x is the variable Quantity that is supposed to be known, let its Fluxion be expressed

by 1, and you will have $\dot{z} = \frac{1}{1+x}$, or $\dot{z} + x\dot{z} - 1 = 0$.

This supposed, let z be expressed by the following Series $ax^m + bx^{m+n} + cx^{m+2n} + dx^{m+3n}$, &c. where a, b, c, d , &c. are Coefficients, and m and n Indexes of Powers, all hereafter to be determined: Then it will always be best to determine the first Term of the Series from the Nature of the Problem the fluxional Equation was drawn from.

Thus

Thus let $x=0$; then it is plain from the Nature of this Area that $z=x$: But when $x=0$, z will be accurately equal to ax^m , all the other Terms of the Series vanishing in respect of this: Therefore, in this Case, $ax^m=x$; and consequently in all others: Therefore $m=1$, and the Coefficient a (if there were any Occasion to determine it here) is also equal to 1.

Next to determine n the Difference of the *Indexes*; let us resume the fluxional Equation before found viz. $\dot{z}+x\dot{z}-1=0$, and since $z=ax^m$, $\mathcal{E}c$. we have $\dot{z}=amx^{m-1}$, and $x\dot{z}=amx^m$: Therefore since we have here two different Powers of x , viz. amx^{m-1} , and amx^m , and since 1 is the Difference of the *Indexes* of these Powers, I make $n=1$ in order to preserve the Form of the Series; as will abundantly be shewn by this, and the following Operations. So that at last the Series is determined to be of the following Form, viz. $z=ax+bx^2+cx^3+dx^4+ex^5+fx^6$, $\mathcal{E}c$. and the Coefficients will be determined by the fluxional Equation, as follows:

$$\left\{ \begin{array}{l} \dot{z} = a + 2bx + 3cx^2 + 4dx^3 \\ \quad + 5ex^4 + 6fx^5. \\ +x\dot{z} = * ax + 2bx^2 + 3cx^3 \\ \quad + 4dx^4 + 5ex^5. \\ -1 = -1 + 0 + 0 + 0 + 0 + 0. \end{array} \right\} \mathcal{E}c. = 0.$$

Where we have the following Equations $a-1=0$,
 $2b+a=0$, $3c+2b=0$, $4d+3c=0$, $5e+4d=0$,

I 3

6f

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$6f + 5e = 0$, &c. Therefore $a = +1$, $b = -\frac{1}{2}$, $c = +\frac{1}{4}$, $d = -\frac{1}{8}$, $e = +\frac{1}{16}$, $f = -\frac{1}{32}$; and $z = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \&c.$

N. B. The Root z in this Example might also be extracted by the Means of second Fluxions, or first Fluents.

EXAMPLE II.

In the foregoing fluxional Equation let now z be known, and let it be proposed to extract the Root x , or to express x by an infinite Series consisting of the several Powers of z .

Since then in the foregoing Equation $\frac{\dot{x}}{1+x} = \dot{z}$, if we make $\dot{z} = 1$, we shall have $1 + x - \dot{x} = 0$. Let $x = az + bz^2 + cz^3 + dz^4 + ez^5 + fz^6$, &c. for this Series must be of the same Form with the foregoing, otherwise they would not be convertible into one another. This supposed, resuming our fluxional Equation $1 + x - \dot{x} = 0$,

$$\left\{ \begin{array}{l} 1 = 1 + 0 + 0 + 0 + 0. \\ + x = x + az + bz^2 + cz^3 + dz^4 \\ \quad + ez^5. \\ - \dot{x} = -a - 2bz - 3cz^2 - 4dz^3 \\ \quad - 5ez^4 - 6fz^5. \end{array} \right\} \&c. = 0.$$

Whence we have the following Equations;
 $1 - a = 0$, $a - 2b = 0$, $b - 3c = 0$, $c - 4d = 0$,

$d - 5e = 0$, $e - 6f = 0$: That is, $1 = a$, $b = \frac{a}{2}$,

$c =$

$c = \frac{b}{3}, d = \frac{c}{4}, e = \frac{d}{5}, f = \frac{e}{6}, \&c.$ So that at last we have the following Series $x = z + \frac{1}{2}az^2 + \frac{1}{3}bz^3 + \frac{1}{4}cz^4 + \frac{1}{5}dz^5 + \frac{1}{6}ez^6, \&c.$ if the Letters $a, b, c, d, \&c.$ represent only Coefficients; but if they stand for whole Terms, as they come up, then we shall have $x = z + \frac{1}{2}az + \frac{1}{3}bz + \frac{1}{4}cz + \frac{1}{5}dz + \frac{1}{6}ez, \&c.$

EXAMPLE III.

It is required to find any Ark as z from its Sine x , which is supposed to be known, in a Circle whose Radius is 1.

Here the fluxional Equation expressing the Relation between the Ark z and its Sine x is

$$\frac{\dot{x}}{\sqrt{1-xx}} = \dot{z}, \text{ or } \frac{\dot{x}^2}{1-xx} = \dot{z}^2. \text{ Make } \dot{x} = 1, \text{ and}$$

$$\text{we shall have } \frac{1}{1-xx} = \dot{z}^2, \text{ or } 1-xx \times \dot{z}^2 = 1.$$

To render this last Equation yet more simple, throw it into second Fluxions, and we shall have

$$1-xx \times 2\dot{z}\ddot{z} - 2x\dot{z}^2 = 0; \text{ or dividing both Sides by } 2\dot{z}, \text{ we have } 1-xx \times \ddot{z} - x\dot{z} = 0; \text{ or thus,}$$

$\ddot{z} - xx\ddot{z} - x\dot{z} = 0.$ Having thus brought our Equation into the simplest Form, let $z = ax^m + bx^{m+n}, \&c.$ and when $x=0$, z will be equal to x , because every infinitely small Ark is equal to its Sine: Therefore $ax^m = x$, and consequently $a=1$, and $m=1$.

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Now to determine n ; since $\ddot{z} - x \dot{x} \dot{z} - x \dot{z} = 0$, and since $z = a x^m$ nearly, we shall have $\ddot{z} = a m \times m - 1 \times x^{m-2}$, and $-x \dot{x} \dot{z} = -a m \times m - 1 \times x^m$, and $-x \dot{z} = -a m \times x^m$. There are therefore two different Powers of x concerned in our fluxional Equation, viz. x^{m-2} , and x^m ; therefore I make $n = 2$, and the Series will stand in the following Form, viz. $z = a x + b x^3 + c x^5 + d x^7 + e x^9 + f x^{11}$, &c. Therefore

$$\left\{ \begin{array}{l} \ddot{z} = 2 \times 3 b x + 4 \times 5 c x^3 + 6 \times 7 d x^5 \\ \quad + 8 \times 9 e x^7 + 10 \times 11 f x^9. \\ -x \dot{x} \dot{z} = -2 \times 3 b x^3 - 4 \times 5 c x^5 \\ \quad - 6 \times 7 d x^7 - 8 \times 9 e x^9. \\ -x \dot{z} = -a x - 3 b x^3 - 5 c x^5 - 7 d x^7 \\ \quad - 9 e x^9. \end{array} \right\} \text{ \&c. } = 0.$$

Whence we have the following Equations for determining the Coefficients; $2 \times 3 b - a = 0$, $4 \times 5 c - 3 \times 3 b = 0$, $6 \times 7 d - 5 \times 5 c = 0$, $8 \times 9 e - 7 \times 7 d = 0$, $10 \times 11 f - 9 \times 9 e = 0$, &c. that is, $a = 1$, $b = \frac{1 \times 1}{2 \times 3} a$, $c = \frac{3 \times 3}{4 \times 5} b$, $d = \frac{5 \times 5}{6 \times 7} c$, $e = \frac{7 \times 7}{8 \times 9} d$, $f = \frac{9 \times 9}{10 \times 11} e$. So that if x be the Sine of any Ark,

the Ark itself will be $x + \frac{1 \times 1}{2 \times 3} a x^3 + \frac{3 \times 3}{4 \times 5} b x^5 + \frac{5 \times 5}{6 \times 7} c x^7 + \frac{7 \times 7}{8 \times 9} d x^9 + \frac{9 \times 9}{10 \times 11} e x^{11}$, &c. supposing

a, b, c, d , &c. to be Coefficients: But if they be the whole Terms, we shall have $z = x + \frac{1 \times 1}{2 \times 3} a x^3 + \frac{3 \times 3}{4 \times 5} b x^5 + \frac{5 \times 5}{6 \times 7} c x^7 + \frac{7 \times 7}{8 \times 9} d x^9 + \frac{9 \times 9}{10 \times 11} e x^{11}$, &c.

N. B. If x be the Cosine of any Ark, the Fluxion of that Ark will be $\frac{-\dot{x}}{\sqrt{1-xx}}$, and its Square $\frac{+\dot{x}^2}{1-xx}$. So that, the same fluxional Equation that

finds the Ark from its Sine will also find it from its Cosine; but here the Series will have a different Beginning from the Series in the former Case: For if q be a quadrantal Ark, when x is infinitely small, the Ark whereof it is the Sine will be x ; and therefore the Ark whereof it is the Cosine will be $q-x$: Therefore $q-x$ will make the two first Terms of this Series; after which the Series will ascend regularly by Powers of x , whose Indexes will have 2 for their common Difference.

But the Invention of the Ark from the Cosine will best be effected thus: If z be the Ark whose Sine is x ; then $q-z$ will be the Ark whose Cosine is x . But $q-z$ is already determined to be

$$q-x + \frac{1 \times 1}{2 \times 3} a x x + \frac{3 \times 3}{4 \times 5} b x x x + \frac{5 \times 5}{6 \times 7} c x x x, \text{ \&c. supposing } a = -x.$$

EXAMPLE IV.

Let now the Reverse of the former Example be proposed; that is, from the Ark z being given to find its Sine or Cosine x , since we find the same fluxional Equation serves for both.

The fluxional Equation is $\frac{\dot{x}^2}{1-xx} = \dot{z}^2$, or making $\dot{z}=1$, we have $\frac{\dot{x}^2}{1-xx} = 1$, or $1-xx = \dot{x}^2$.

Throw

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Throw this last Equation into second Fluxions, and we have $-2x\dot{x}=2\dot{x}\ddot{x}$, or $-x=\ddot{x}$, or $x+\ddot{x}=0$. Let $x=a z^m+b z^{m+n}$, &c. and first let x be the *Sine* sought: Then when $z=0$, x will be equal to z : Therefore $a z^m=z$: Therefore $m=1$, and $a=1$, as in the former Example. Again, in the fluxional Equation $x+\ddot{x}=0$, since $z=z^m$ setting aside the Coefficient, and $\ddot{x}=z^{m-2}$, we have $n=2$. Let therefore $x=a z+b z^3+c z^5+d z^7$, &c. and the Equations will be as follows:

$$\left. \begin{aligned} x &= a z + b z^3 + c z^5 + d z^7 + e z^9. \\ \ddot{x} &= 2 \times 3 b z + 4 \times 5 c z^3 + 6 \times 7 d z^5 \\ &\quad + 8 \times 9 e z^7 + 10 \times 11 f z^9. \end{aligned} \right\} \text{&c.} = 0.$$

Whence, and from what has been proved before,

$$\text{we have } a=1, b=-\frac{a}{2 \times 3}, c=-\frac{b}{4 \times 5}, d=-\frac{c}{6 \times 7},$$

$$e=-\frac{d}{8 \times 9}, f=-\frac{e}{10 \times 11}, \text{ &c. Let } a, b, c, d, \text{ &c.}$$

represent the several Terms of the Series; then we

$$\text{have } x=z-\frac{1}{2 \times 3} a z^2-\frac{1}{4 \times 5} b z^2-\frac{1}{6 \times 7} c z^2-\frac{1}{8 \times 9} d z^2-\frac{1}{10 \times 11} e z^2, \text{ &c.}$$

If x be the *Cosine* sought, then, when $z=0$, the Cosine x will be 1, viz. the Radius: Therefore the first Term, viz $a z^m=1$; therefore $a=1$, and $m=0$: And since $n=2$, as before, let $x=a+b z^2+c z^4+d z^6+e z^8+f z^{10}$, &c. and the Equations will be as follow:

$$x=a$$

$$\left\{ \begin{aligned} x &= a + bz^2 + cz^4 + dz^6 + ez^8. \\ \dot{x} &= 1 \times 2 b + 3 \times 4 c z^2 + 5 \times 6 d z^4 + 7 \times 8 e z^6 + 9 \times 10 f z^8. \end{aligned} \right\} \text{Et c.} = 0.$$

Whence, and from that $a=1$, we have the following

Equations; $a=1$, $b=-\frac{a}{1 \times 2}$, $c=-\frac{b}{3 \times 4}$,

$d=-\frac{c}{5 \times 6}$, $e=-\frac{d}{7 \times 8}$, $f=-\frac{e}{9 \times 10}$, Et c. Make

$a, b, c, d, \text{Et c.}$ represent the Terms of the Series;

then we shall have $x=1-\frac{1}{1 \times 2} a z^2-\frac{1}{3 \times 4} b z^2-\frac{1}{5 \times 6} c z^2-\frac{1}{7 \times 8} d z^2-\frac{1}{9 \times 10} e z^2, \text{Et c.}$

EXAMPLE V.

In a Circle whose Radius is 1, let x and z be the Sines or Cosines of two Arks, whereof the former Ark is to the latter as 1 to p a constant Quantity: It is required having given x to find z .

Now because the Proportion of these two Arks is that of 1 to p , let the true Quantities be what they will, it follows, that if these Quantities be made to vary, their Fluxions also must be to one another as 1 to p , and their Squares as 1 to p^2 . But the Square of the Fluxion of the Ark whose

Sine or Cosine is x is $\frac{\dot{x}^2}{1-x x}$, or $\frac{1}{1-x x}$, if $\dot{x}$ be made equal to 1; and the Square of the Fluxion

of the Ark whose Sine or Cosine is z is $\frac{\dot{z}^2}{1-z z}$:

Whence

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Whence we have this Proportion $\frac{1}{1-xx}$ to $\frac{\dot{z}^2}{1-zz}$ as 1 to p^2 , and this Equation $p^2 - p^2 z^2 = \dot{z}^2 - x^2 \dot{z}^2$. Throw this Equation into second Fluxions, and you will have $-2p^2 z \dot{z} = 2 \dot{z} \ddot{z} - 2 x \ddot{x} \dot{z} - 2 x \dot{x} \dot{z}$. Divide all by $2 \dot{z}$, and you will have $-p^2 z = \ddot{z} - x \ddot{x} - x \dot{x}$, or $\ddot{z} - x \ddot{x} - x \dot{x} + p^2 z = 0$.

Let $z = ax^m + bx^{m+n}$, &c. and first let x and z be Sines: Then it is plain that when x is infinitely small, the Ark whereof it is the Sine will also be equal to x ; and therefore the Ark whereof z is the Sine will be px . And since px as well as x must be infinitely small, we shall have the Sine $z = px$, and consequently $ax^m = px$; whence $a = p$, and $m = 1$. Again, in the fluxional Equation $\ddot{z} - x \ddot{x} - x \dot{x} + p^2 z = 0$, $\ddot{z}$ will be as x^{m-2} ; $x \ddot{x}$, and $x \dot{x}$, and $p^2 z$ will all be as x^m ; wherefore $n = 2$; Therefore $z = ax + bx^3 + cx^5 + dx^7 + ex^9 + fx^{11}$, &c. Whence we have

$$\left\{ \begin{array}{l} +\ddot{z} = +2 \times 3b x + 4 \times 5c x^3 + 6 \times 7d x^5 \\ \quad + 8 \times 9e x^7 + 10 \times 11f x^9. \\ -x \ddot{x} = * -2 \times 3b x^3 - 4 \times 5c x^5 \\ \quad - 6 \times 7d x^7 - 8 \times 9e x^9. \\ -x \dot{x} = -a x - 3b x^3 - 5c x^5 - 7d x^7 \\ \quad - 9e x^9. \\ +p^2 z = +p^2 a x + p^2 b x^3 + p^2 c x^5 \\ \quad + p^2 d x^7 + p^2 e x^9. \end{array} \right\} \text{&c.} = 0.$$

Whence, and from what has before been proved that $a = p$, we have the following Equations,

$a = p,$

$$a=p, b=\frac{1 \times 1 - p^2}{2 \times 3} a, c=\frac{3 \times 3 - p^2}{4 \times 5} b, d=\frac{5 \times 5 - p^2}{6 \times 7} c,$$

$$e=\frac{7 \times 7 - p^2}{8 \times 9} d, f=\frac{9 \times 9 - p^2}{10 \times 11} e, \text{ \&c.} \text{ Whence if}$$

$a, b, c, d, \text{ \&c.}$ instead of Coefficients stand for the Terms of the Series, we shall have $z = px +$

$$\frac{1 \times 1 - p^2}{2 \times 3} a x^2 + \frac{3 \times 3 - p^2}{4 \times 5} b x^2 + \frac{5 \times 5 - p^2}{6 \times 7} c x^2 +$$

$$\frac{7 \times 7 - p^2}{8 \times 9} d x^2 + \frac{9 \times 9 - p^2}{10 \times 11} e x^2, \text{ \&c.} \text{ Where it may}$$

be observed that if p be an odd Number, the Series will break off, and so be finite; otherwise it will run on *ad infinitum*.

Before we can apply our Manner of Reasoning to *Cosines*, we are obliged to premise the following Lemma.

LEMMA. Fig. 25.

In a Circle whose Radius is 1, let q be a quadrantal Ark, let x be any other Ark infinitely small, and let p be any affirmative whole Number: I say then, that the Cosine of the Ark $pq - px$, or of the Ark $q - x \times p$ will be ± 1 , or $\pm px$: That is, If p be an *even* Number, the Cosine will be ± 1 , according as the Number p is what they call *paritèr par* or otherwise: But if p be an *odd* Number, then the Cosine will be $\pm px$, according as the Number $p - 1$, is *paritèr par* or otherwise.

This will best appear by dividing the Circle $a b c d$ whose Radius is 1 into four equal Quadrants

$a b,$

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ab, bc, cd, da , and drawing the cross Diameters ac and bd : For then if we call all the Cosines that fall within the first and last Quadrants ab and ad affirmative, we shall have the Cosine of $2q-2x=-1$, that of $4q-4x=+1$, that of $6q-6x=-1$, that of $8q-8x=+1$; and so on alternately: We shall have also the Cosine of $q-x=+x$, that of $3q-3x=-3x$, that of $5q-5x=+5x$, that of $7q-7x=-7x$; and so on alternately *ad infinitum*.

This premised, let now x and z be Cosines; and let x be infinitely small: Then it is plain that the Ark. whereof x is the Cosine will be qx ; therefore the Ark whereof z is the Cosine will be $pq-px$. But the Cosine of the Ark $pq-px=+px$, or ± 1 , by the *Lemma*; therefore z or $ax^m=\pm px$, or ± 1 , according to the Conditions already laid down in the *Lemma*. Now if $z=\pm px$, the same Series that was before computed for the Sines will also serve for the Cosines, making $a=\pm p$, as Occasion requires. But if z , or $ax^m=\pm 1$; then we shall have $a=\pm 1$, and $m=0$, and a new Series must be drawn out of the fluxional Quantities by making $z=a+bx^2+cx^4+dx^6+ex^8+fx^{10}$, &c. as follows:

$$z=$$

$$\left[\begin{array}{l} z = 1 \times 2 b + 3 \times 4 c x^2 + 5 \times 6 d x^4 \\ \quad + 7 \times 8 e x^6 + 9 \times 10 f x^8. \\ - x \dot{z} = * - 1 \times 2 b x^2 - 3 \times 4 c x^4 \\ \quad - 5 \times 6 d x^6 - 7 \times 8 e x^8. \\ - x \ddot{z} = * - 1 \times 2 b x^2 - 4 c x^4 - 6 d^6 \\ \quad - 8 e x^8. \\ + p^2 z = + p^2 a + p^2 b x^2 + p^2 c x^4 \\ \quad + p^2 d x^6 + p^2 e x^8. \end{array} \right] \mathcal{E}c. = 0.$$

From these Equations deducing the Coefficients $a, b, c, d, \mathcal{E}c.$ and then making them stand for whole Terms, the new Series will stand as follows:

$$z = +1 + \frac{0 \times 0 - p^2}{1 \times 2} a x^2 + \frac{2 \times 2 - p^2}{3 \times 4} b x^2 + \frac{4 \times 4 - p^2}{5 \times 6} c x^2 + \frac{6 \times 6 - p^2}{7 \times 8} d x^2 + \frac{8 \times 8 - p^2}{9 \times 10} e x^2, \mathcal{E}c. \text{ Therefore the Series relating to the Cosines always break off.}$$

N. B. 0 must be reckoned amongst the Numbers that are called *paritèr par.*

EXAMPLE VI.

To throw the following Quantity $\overline{p+qx}^m$ into a Series finite or infinite, as $a x^r + b x^r + \mathcal{E}c.$

Make $\overline{p+qx}^m = z$, and making $\dot{x} = 1$, we shall

$$\text{have } \ddot{z} = m q \times \overline{p+qx}^{m-1} = m q \times \frac{\overline{p+qx}^m}{\overline{p+qx}} = \frac{m q z}{\overline{p+qx}}.$$

Therefore $p \ddot{z} + q x \ddot{z} - m q z = 0$. Moreover since $\overline{p+qx}^m$

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$\overline{p+qx}^m = z$, when $x=0$ we have $z=p^m$: Therefore $a x^r = p^m$: Therefore $a=p^m$, and $r=0$. Again, In the fluxional Equation $p\dot{z} + qx\dot{z} - mqz = 0$ we have $p\dot{z}$ as x^{r-s} ; and $qx\dot{z}$, and mqz will both be as x^r . Therefore $s=1$, and $z = a + bx + cx^2 + dx^3 + ex^4 + fx^5$, &c. Whence we have as follows:

$$\left\{ \begin{array}{l} p\dot{z} = pb + 2pcx + 3pd x^2 + 4pe x^3 \\ \quad + 5pf x^4. \\ + qx\dot{z} = * + qbx + 2qc x^2 \\ \quad + 3qd x^3 + 4qe x^4. \\ -mqz = -mq a - mqbx - mqcx^2 \\ \quad -mqd x^3 - mqe x^4. \end{array} \right\} \text{ &c. } = 0.$$

Hence making $a=p^m$ as above, and determining the other Coefficients, as usual, and then making them stand for whole Terms, we shall have

$$\overline{p+qx}^m = p^m + \frac{m}{1} \times \frac{aqx}{p} + \frac{m-1}{2} \times \frac{bqx}{p} + \frac{m-2}{3} \times \frac{cqx}{p} + \frac{m-3}{4} \times \frac{dqx}{p} + \frac{m-4}{5} \times \frac{eqx}{p}, \text{ &c.}$$

The

*The Method of computing the FORMS in the
second Part of the LOGOMETRIA.*

PROBLEM I.

LEMMA.

To find the Fluent of this Fluxion $\frac{d\dot{T}}{T \pm R}$, where d
and R are constant, and T variable.

Make $T \pm R = x$, and we have $\dot{T} = \dot{x}$, and
 $\frac{d\dot{T}}{T \pm R} = \frac{d\dot{x}}{x}$, whose Fluent is the Measure of the
Ratio of x to R , or of x to any other constant
Quantity, *ad Modulum* d^* : Therefore the Fluent
of $\frac{d\dot{T}}{T \pm R}$ is the Measure of the Ratio of $T \pm R$ to
 R *ad Modulum* d . Q. E. I.

PROBLEM II.

LEMMA.

To find the Fluent of this Fluxion $\frac{dR^2 \dot{T}}{SS}$, where d
and R are constant, T variable, and $SS = T^2 - R^2$.

Now $\frac{dR^2 \dot{T}}{SS}$ or $\frac{dR^2 \dot{T}}{T^2 - R^2} = \frac{\frac{1}{2} dR \dot{T}}{T - R} - \frac{\frac{1}{2} dR \dot{T}}{T + R}$,
as will be evident by bringing these two Fractions
into one. But, by the first Problem, the Fluent

* By Prop. i. Coroll. i. Logometriae.

of this latter Fluxion is the Measure of the Ratio of $T+R$ to R , *ad Modulum* $-\frac{1}{2}dR$; and the Fluent of the former Fluxion is the Measure of the Ratio of $T-R$ to R *ad Modulum* $\frac{1}{2}dR$, or of R to $T-R$, *ad Modulum* $-\frac{1}{2}dR$. Now as these Measures belong to the same *Modulus*, and consequently to the same *System*, if put together they will constitute the Measure of the two Ratio's put together, or of the Ratio of $T+R$ to $T-R$.

Wherefore the Fluent of $\frac{dR^2 \dot{T}}{SS}$ or of $\frac{\frac{1}{2}dR \dot{T}}{T-R} - \frac{\frac{1}{2}dR \dot{T}}{T+R}$ is the Measure of the Ratio of $T+R$ to

$T-R$ *ad Modulum* $-\frac{1}{2}dR$, or it is the Measure of half that Ratio *ad Modulum* $-dR$. But half the Ratio of $T+R$ to $T-R$ is the Ratio of $T+R$ to S , since by Hypothesis $SS=TT-RR=$

$\overline{T+R} \times \overline{T-R}$: Wherefore the Fluent of $\frac{dR^2 \dot{T}}{SS}$ is found at last to be the Measure of the Ratio of $T+R$ to S *ad Modulum* $-dR$. Q. E. I.

PROBLEM III.

LEMMA Fig. 26.

To find the Fluent of this Fluxion $\frac{dR^2 \dot{T}}{SS}$, where d and R are constant, T variable, and $SS=TT+RR$.

Make the Line $AB=R$ the Radius, $BC=T$ the Tangent, and $ADC=S$ the Secant of the Arc BD ;

BD; and produce BC to c , so that Cc may be equal to T , and draw Ac cutting the Arc BD produced in d , as also CE perpendicular to AEc : Then the similar Triangles ADd , ACE , as also CcE , AcB or ACB will give the following Proportions, viz. Dd to CE as AD or AB to AC ; and CE to Cc as AB to AC , and consequently Dd to Cc as $\overline{AB}^2$ to $\overline{AC}^2$. Therefore

$$Dd = Cc \times \frac{\overline{AB}^2}{\overline{AC}^2} = \frac{R^2 \dot{T}}{S S}. \text{ But } Dd \text{ is the Fluxion}$$

of the Arc BD, which is the Measure of the Angle BAC *ad Radium vel Modulum* AB. Therefore

the Fluent of $\frac{R^2 \dot{T}}{S S}$ is the Measure of an Angle,

whose Radius, Tangent, and Secant are R , T and S

respectively: Therefore the Fluent of $\frac{dR^2 \dot{T}}{S S}$ is the

Measure of the same Angle *ad Modulum* dR ,

that is, it is the Measure of an Angle, whose

Radius, Tangent and Secant are as R , T , and S

respectively, but whose true Radius is dR .

Q. E. I.

PROBLEM IV.

To find the Fluent of this Fluxion in the first FORM $\frac{d\dot{z}z^{n-1}}{e+fz^n}$; where, as well as in all the Problems that follow, $d, e, f,$ and $n,$ are constant, and z variable.

Make $e+fz^n=x$, and we shall have $\eta f\dot{z}z^{n-1}=\dot{x}$, and $\frac{\eta f\dot{z}z^{n-1}}{e+fz^n}=\frac{\dot{x}}{x}$, and $\frac{d\dot{z}z^{n-1}}{e+fz^n}=\frac{d}{\eta f}\times\frac{\dot{x}}{x}$; whose Fluent is the Measure of the Ratio of x to e , or of $e+fz^n$ to e ad Modulum $\frac{d}{\eta f}$; and is usually expressed thus $\frac{d}{\eta f}\bigg|\frac{e+fz^n}{e}$. Q. E. I.

PROBLEM V.

To find the Fluent of this Fluxion in the second FORM

$$\frac{d\dot{z}z^{\frac{1}{2}n-1}}{e+fz^n}.$$

SOLUTION I.

Make $-\frac{e}{f}=RR$, or $\frac{e}{f}=-RR$, $z^n=T^2$, and $\frac{e}{f}+z^n$, or $\frac{e+fz^n}{f}$, or $T^2-R^2=S^2$, and we shall have $z^{\frac{1}{2}n}=T$, and $\frac{1}{2}\eta\dot{z}z^{\frac{1}{2}n-1}=\dot{T}$. Multiply one Side into $\frac{e}{e+fz^n}$, and the other into its Equal $-\frac{R^2}{S^2}$, and we shall have $\frac{\frac{1}{2}\eta e\dot{z}z^{\frac{1}{2}n-1}}{e+fz^n}=-\frac{R^2\dot{T}}{S^2}$. Again, multiply

multiply both Sides into $\frac{2}{\eta e} d$, and we shall have

$$\frac{d \dot{z} z^{\frac{1}{\eta}-1}}{e+fz^{\eta}} = \frac{2}{\eta e} \times -\frac{dR^2 T}{S^2}; \text{ whose Fluent } \frac{2 dR}{\eta e} \left| \frac{R+T}{S} \right|$$

is the Fluent sought by Prob. II. Q. E. I.

SOLUTION II.

Make $\frac{e}{f} = R^2$, and $z^{\eta} = T^2$; also $\frac{e}{f} + z^{\eta}$, or $\frac{e+fz^{\eta}}{f}$, or $R^2 + T^2 = S^2$, and we shall then have

$$\frac{d \dot{z} z^{\frac{1}{\eta}-1}}{e+fz^{\eta}} = \frac{2}{\eta e} \times \frac{dR^2 T}{S^2}; \text{ whose Fluent, by the third}$$

Lemma, is the Measure of an Angle, whose *Radius*,

Tangent, and *Secant* are as R, T, and S, respectively, but whose true *Radius* is $\frac{2}{\eta e} dR$. Q. E. I.

Whence it appears,

First, That this Fluent, and consequently all others that depend upon it, has two different Values, one *logometrical*, and the other *trigonometrical*.

Secondly, That one of these Values will always be possible, but that they can never both be possible at once; and consequently that whenever one fails, Recourse must be had to the other: For if the Equation $-\frac{e}{f} = R^2$ in the *logometrical* Expression be possible, then the other Equation $\frac{e}{f} = R^2$ in the *trigonometrical* one will be impossible; and *vice versa*.

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Tbirdly, That though this Fluent, as well as all others that depend upon it, is only expressed in the *logometrical* Style, yet when this fails, the *trigonometrical* Fluent will be easily had by changing the Sign of R^2 , and so making R , as well as the *Modulus* that depends upon it, possible; and then making R , T , and S , instead of expressing the Terms of a *Ratio*, to express the Proportion between the *Radius*, *Tangent*, and *Secant* of an Angle, whose Measure, according to this new corrected *Modulus*, will be the Fluent sought.

PROBLEM VI.

To continue the Series's of Fluents belonging to the first and second FORMS, from the Fluents found in the fourth and fifth Problems, upwards ad infinitum.

The general Characteristic of the two first *Forms* is $\frac{d\dot{z} z^{\theta-1}}{e+fz^n}$, where θ is either a whole Number as in the *first Form*, or a Fraction whose Denominator is 2, as in the *Second*, and answers to the Author's $\theta + \frac{1}{2}$.

Let $\frac{d\dot{z} z^{\theta-1}}{e+fz^n} = \dot{a}$, and the Fluxion next above it, viz. $\frac{d\dot{z} z^{\theta+n-1}}{e+fz^n}$, or $z^n \dot{a} = \dot{b}$; and we shall have, from the *first* Equation, $d\dot{z} z^{\theta-1} = e\dot{a} + fz^n \dot{a} = e\dot{a} + f\dot{b}$ by the *Second*; and if we take the Fluents on both Sides, and suppose them equal, we shall have

have $\frac{d}{\theta \eta} z^{\theta \eta} = e a + f b$; whence $b = \frac{d}{\theta \eta f} z^{\theta \eta} - \frac{e a}{f}$;
by the Means of which Equation, if the Fluent a
be known, the next Fluent above it may be found;
and then making this equal to a , the next above
this will also be found; and so on *ad infinitum*.
Q. E. F.

EXAMPLE I.

Let $\theta = 1$, and consequently $\frac{d}{\theta \eta f} z^{\theta \eta} = \frac{d}{\eta f} z^{\eta}$,
and we shall have $a = \frac{d}{\eta f} \left| \frac{e + f z^{\eta}}{e} \right|$ according to the
fourth Problem, and $\frac{e a}{f} = \frac{d e}{\eta f f} \left| \frac{e + f z^{\eta}}{e} \right|$, and $\frac{d}{\theta \eta f}$
 $z^{\theta \eta} - \frac{e a}{f}$, or b , or the Fluent of the Fluxion
 $\frac{d \dot{z} z^{2\eta-1}}{e + f z^{\eta}} = \frac{d}{\eta f} z^{\eta} - \frac{d e}{\eta f f} \left| \frac{e + f z^{\eta}}{e} \right|$: And if now we
make this last Fluent equal to a , and $\theta = 2$, we
shall find after the same Manner the Fluent of
 $\frac{d \dot{z} z^{3\eta-1}}{e + f z^{\eta}}$; and so on *ad infinitum*.

EXAMPLE II.

Let $\theta = \frac{1}{2}$, and we shall have $\frac{d}{\theta \eta f} z^{\theta \eta} = \frac{2 d}{\eta f} z^{\frac{1}{2} \eta}$,
and $a = \frac{2}{\eta e} d R \left| \frac{R + T}{S} \right|$, and $\frac{e a}{f} = \frac{2}{\eta f} d R \left| \frac{R + T}{S} \right|$,
and $\frac{d}{\theta \eta f} z^{\theta \eta} - \frac{e a}{f}$, or b , or the Fluent of $\frac{d \dot{z} z^{\frac{1}{2} \eta - 1}}{e + f z^{\eta}}$,
 $= \frac{2 d z^{\frac{1}{2} \eta}}{\eta f} - \frac{2}{\eta f} d R \left| \frac{R + T}{S} \right|$.

PROBLEM VII.

To continue the Series's of Fluents belonging to the first, and second FORMS, from the Fluents found in the fourth and fifth Problems, downwards ad infinitum.

This may be done by the Reverse of the sixth Problem, that is, by supposing b known, and a sought; in which Case a will be equal to $\frac{d}{\theta \eta e} z^{\theta \eta}$ $-\frac{fb}{e}$: But because there are some Cases wherein a Descent of this Nature from Fluent to Fluent cannot without some Difficulty be made, I shall rather propose the following Method, which is much easier and more expeditious.

This Fluxion of the first Form $\frac{d \dot{z} z^{\lambda \eta - 1}}{e + f z^{\eta}}$, where $\theta = \lambda$, by changing the Sign of η , and changing the Quantities e and f for one another, becomes $\frac{d \dot{z} z^{-\lambda \eta - 1}}{f + e z^{\eta}} =$ (by multiplying the Terms of the Fraction into z^{η} , which does not affect its Value) $\frac{d \dot{z} z^{-\lambda \eta + \eta - 1}}{e + f z^{\eta}}$; which is a Fluxion of the same first Form, whose Fluent has not yet been computed, and where $\theta = 1 - \lambda$. In like Manner this Fluxion of the second Form $\frac{d \dot{z} z^{-\lambda \eta + \frac{1}{2} \eta - 1}}{e + f z^{\eta}}$, wherein the Author's $\theta = \lambda$; this Fluxion, I say, will by a like

like Change be turned into this $\frac{d\dot{z} z^{-\lambda + \frac{1}{2}\eta - 1}}{e + fz^\eta}$, another Fluxion of the same *Form*, whose Fluent has not yet been computed, and where $\theta = -\lambda$. But if this be the Case of the Fluxions, it must also be true of the Fluents themselves; that is, Every Fluent of the *first Form* wherein $\theta = \lambda$ may, by changing the Sign of η , and the Quantities e and f for one another become a Fluent of the same *Form*, wherein $\theta = 1 - \lambda$; and every Fluent of the *second Form*, wherein $\theta = \lambda$ is by this Means convertible into another of the same *Form* wherein $\theta = -\lambda$. And thus may all the Fluents of the two first *Forms*, which have not yet been found, be with Ease computed.

EXAMPLE I.

The Fluent of this Fluxion of the *first Form* $\frac{d\dot{z} z^{\eta-1}}{e + fz^\eta}$, wherein $\theta = 1$ is $\frac{d}{\eta f} \left| \frac{e + fz^\eta}{e} \right|$: Therefore the Fluent of this Fluxion of the same *Form* $\frac{d\dot{z}}{ze + fz^\eta}$, where $\theta = 0$ or $1 - 1$, is $\frac{-d}{\eta e} \left| \frac{f + ez^{-\eta}}{f} \right|$
 $= -\frac{d}{\eta e} \left| \frac{e + fz^\eta}{fz^\eta} \right|$.

EXAMPLE II.

If $R = \sqrt{\frac{e}{f}}$, $T = z^{\frac{1}{2}\eta}$, $S = \sqrt{\frac{e + fz^\eta}{f}}$; then the Fluent of this Fluxion of the *second Form*,
 $d\dot{z} z$

$\frac{d\dot{z} z^{\frac{1}{2}n-1}}{e+fz^n}$, where $\theta=1$ will be $\frac{2d}{nf} z^{\frac{1}{2}n} - \frac{2}{nf} dR$

$\left| \frac{R+T}{S} \right|$: Therefore if $r = \sqrt{\frac{-f}{e}}$, $t = z^{-\frac{1}{2}n}$ or $\frac{1}{z^{\frac{1}{2}n}}$

and $s = \frac{\sqrt{f+ez^{-n}}}{e}$, or $\frac{\sqrt{e+fz^n}}{ez^n}$; then the Fluent

of this Fluxion of the same Form $\frac{d\dot{z} z^{-\frac{1}{2}n-1}}{e+fz^n}$,

where $\theta=-1$, will be $\frac{-2d}{ne z^{\frac{1}{2}n}} + \frac{2}{ne} dr \left| \frac{r+t}{s} \right|$.

After the same Manner may the Fluent of $\frac{d\dot{z} z^{-n-1}}{e+fz^n}$ of the *first Form* be computed from the Fluent of $\frac{d\dot{z} z^{2n-1}}{e+fz^n}$ of the same *Form*; and the Fluent of $\frac{d\dot{z} z^{-\frac{3}{2}n-1}}{e+fz^n}$ of the *second Form* from the Fluent of $\frac{d\dot{z} z^{\frac{1}{2}n-1}}{e+fz^n}$ of the same *Form*, and so on *ad infinitum*.

SCHOLIUM.

From what has been delivered it plainly appears, that the Fluent of this Fluxion of the second Form $\frac{d\dot{z} z^{\frac{1}{2}n-1}}{e+fz^n}$, where $\theta=0$, is convertible into itself; that is, supposing all Things as in the second *Example*, the Fluent of this Fluxion will be either $\frac{2}{ne} dR \left| \frac{R+T}{S} \right|$, or $\frac{-2}{nf} dr \left| \frac{r+t}{s} \right|$.

PRO-

PROBLEM VIII.

To find the Fluent of this Fluxion of the fifth FORM

$$\frac{d\dot{z}}{z\sqrt{e+fz^n}}.$$

Let $\sqrt{e+fz^n}=P$, and we have $e+fz^n=P^2$,
and $z^n=\frac{P^2-e}{f}$, and $\eta\dot{z}z^{n-1}=\frac{2}{f}\dot{P}\sqrt{e+fz^n}$, and

$$\frac{\eta\dot{z}z^{n-1}}{\sqrt{e+fz^n}}=\frac{2}{f}\dot{P}, \text{ and } \frac{d\dot{z}z^{n-1}}{\sqrt{e+fz^n}}=\frac{2}{\eta f}\dot{P}.$$

Divide the first Side by z^n , and the other by its Equal $\frac{P^2-e}{f}$, and we shall have $\frac{d\dot{z}z^{n-1}}{\sqrt{e+fz^n}}$, or $\frac{d\dot{z}}{z\sqrt{e+fz^n}}$,

$$=\frac{2d}{\eta}\frac{\dot{P}}{P^2-e} \text{ a Fluxion of the second Form, where}$$

$\eta=0$, and whose Fluent, *mutatis mutandis*, will be the Fluent sought; that is, if for d in the second

Form, we write $\frac{2d}{\eta}$; for z , P ; for η , 2 ; for e , $-e$;

and for f , $+1$; and moreover, if we make $R=\sqrt{e}$,
 $T=P$ or $\sqrt{e+fz^n}$, and $S=\sqrt{P^2-e}$, or $\sqrt{fz^n}$;

the Fluent sought will be $\frac{-2}{\eta e}dR\left|\frac{R+T}{S}\right|$.

Q. E. I.

PROBLEM IX.

To continue the Series of Fluents belonging to the fifth FORM, from the Fluent found in the eighth Problem, upwards and downwards ad infinitum.

Let $\frac{d\dot{z} z^{\theta-1}}{\sqrt{e+fz^{\theta}}} = \dot{a}$, and $\frac{d\dot{z} z^{\theta+1-1}}{\sqrt{e+fz^{\theta}}} = \dot{b}$, or $z^{\theta} \dot{a} = \dot{b}$;

let also $\sqrt{e+fz^{\theta}} = P$, and $z^{\theta} = N$; and from this last Equation we have $\theta \eta \dot{z} z^{\theta-1} = \dot{N}$, and $\frac{\theta \eta d\dot{z} z^{\theta-1}}{\sqrt{e+fz^{\theta}}} = \frac{d\dot{N}}{P}$. Multiply the first Side

by $e+fz^{\theta}$, and the other by its Equal P^2 , and we shall have $\theta \eta e \dot{a} + \theta \eta f z^{\theta} \dot{a}$, or $\theta \eta e \dot{a} + \theta \eta f \dot{b} = d\dot{N} P$, which I shall call the Fluxion of the first Part of the general Equation. Again, since $e+fz^{\theta} = P^2$, we have $\eta f \dot{z} z^{\theta-1} = 2 \dot{P} P$, and $\frac{1}{2} \eta f \dot{z} z^{\theta-1} = \dot{P} P$, and $\frac{\frac{1}{2} \eta f \dot{z} z^{\theta-1}}{\sqrt{e+fz^{\theta}}} = \dot{P}$, and $\frac{\frac{1}{2} \eta f d\dot{z} z^{\theta+1-1}}{\sqrt{e+fz^{\theta}}} = \dot{P}$, or $\frac{1}{2} \eta f \dot{b} = d\dot{N} \dot{P}$,

which I call the Fluxion of the second Part of the general Equation. Join this with the first Part, and we have $\theta \eta e \dot{a} + \theta \eta + \frac{1}{2} \eta \times f \dot{b} = d\dot{N} P + d\dot{N} \dot{P}$, and if we make the Fluents on both Sides equal,

we have $\theta \eta e a + \theta \eta + \frac{1}{2} \eta \times f b = d\dot{N} P = z^{\theta} dP$; and

consequently $\theta e a + \theta + \frac{1}{2} \times f b = \frac{z^{\theta}}{\eta} dP$; by the

Means of which Equation a and b may either of them be found when the other is known.
Q. E. F.

EXAMPLE

EXAMPLE I.

Let the Fluent of $\frac{d\dot{z}}{z\sqrt{e+fz^n}}=b$ be given, and let that of $\frac{d\dot{z}z^{n-1}}{\sqrt{e+fz^n}}=a$ be sought.

Here $\theta=-1$, and the general Equation, when applied to this particular Case gives $-ea-\frac{1}{2}fb=\frac{-1}{\eta z^n}dP$:
 $=\frac{z^{-n}}{\eta}dP=\frac{1}{\eta z^n}dP$, whence $ea+\frac{1}{2}fb=\frac{-1}{\eta z^n}dP$:

But $b=\frac{-2}{\eta e}dR\left|\frac{R+T}{S}\right|$ by the eighth Problem;
 wherefore $ea-\frac{fd}{\eta e}R\left|\frac{R+T}{S}\right|=-\frac{1}{\eta z^n}dP$, and
 $a=\frac{-1}{\eta e z^n}dP+\frac{f}{\eta e^2}dR\left|\frac{R+T}{S}\right|$.

EXAMPLE II.

If the Fluent of $\frac{d\dot{z}}{z\sqrt{e+fz^n}}$, where $\theta=0$, be supposed equal to a , and that of $\frac{d\dot{z}z^{n-1}}{\sqrt{e+fz^n}}=b$ be sought; then will the Quantity θea vanish out of the general Equation, when applied to this particular Case; and the other Terms will be $\frac{1}{2}fb=\frac{1}{\eta}dP$, and b or the Fluent of $\frac{d\dot{z}z^{n-1}}{\sqrt{e+fz^n}}=\frac{2}{\eta f}dP$.

PRO-

PROBLEM X.

To compute any Number of Fluents at pleasure of the sixth FORM.

The Fluents of this Form are derived from those of the Fifth; for any Fluent of the fifth Form, where $\theta = \lambda$, by changing the Sign of η , and the Quantities e and f for one another, is converted into a Fluent of the sixth Form, where

$\theta = -\lambda$: For $\frac{d\dot{z} z^{\lambda-1}}{\sqrt{e+fz}}$ by this Means becomes $\frac{d\dot{z} z^{-\lambda-1}}{\sqrt{f+ez}}$, that is, (by multiplying the Numerator into $z^{\frac{1}{2}}$, and the Denominator into $\sqrt{z}$) $\frac{d\dot{z} z^{-\lambda+\frac{1}{2}}}{\sqrt{e+fz}}$. Q. E. D.

PROBLEM XI.

To compute any Number of Fluents at pleasure of the third FORM.

Call any Fluent of the third Form where $\theta = \lambda$, c , and call the two Fluents of the fifth Form, where $\theta = \lambda$, and $\lambda + 1$, a and b respectively; and we shall always have $ea + fb = c$. For since by

Hypothesis $\frac{d\dot{z} z^{\lambda-1}}{\sqrt{e+fz}} = a$, if we multiply both Sides by $e+fz$, we shall have $\frac{d\dot{z} z^{\lambda-1}}{\sqrt{e+fz}} \times \sqrt{e+fz}$,

or $d\dot{z} z^{\lambda-1} \times \sqrt{e+fz}$, or $i = ea + fb$; whence $ea + fb = c$. Q. E. D.

PRO-

PROBLEM XII.

To find any Number of Fluents at pleasure of the fourth FORM.

This is done by observing that every Fluent of the third Form, where $\theta = \lambda$ is, by changing the Sign of η and the Quantities e and f for one another, converted into a Fluent of the fourth Form, where $\theta = -\lambda - 1$, and every Fluent of the third Form, where $\theta = -\lambda$ is converted into a Fluent of the fourth Form, where $\theta = +\lambda - 1$: For $d\dot{z} z^{\lambda-1} \times \sqrt{e+fz^n}$ by this Means will be changed into $d\dot{z} z^{-\lambda-1} \times \sqrt{f+ez^{-n}}$, that is, (by dividing one of the Factors, viz. $d\dot{z} z^{-\lambda-1}$ by $z^{\frac{1}{2}n}$, and multiplying the other, viz. $\sqrt{f+ez^{-n}}$ into $\sqrt{z^n}$, which cannot affect the Product) $d\dot{z} z^{-\lambda-\frac{1}{2}n-1} \times \sqrt{e+fz^n}$, or $d\dot{z} z^{-\lambda-\eta-\frac{1}{2}n-1} \times \sqrt{e+fz^n}$. Q. E. D.

PROBLEM XIII.

To find the Fluents of this Fluxion of the ninth

$$\text{FORM } \frac{d\dot{z} z^{n-1}}{g+bz^n \times \sqrt{e+fz^n}}.$$

Let $\sqrt{e+fz^n} = P$, and we shall have $e+fz^n = P^2$, $z^n = \frac{P^2 - e}{f}$, and $g+bz^n = \frac{fg - eb + bP^2}{f}$: We

shall have moreover $\eta \dot{z} z^{n-1} = \frac{2}{f} P \dot{P}$, and $\frac{\eta \dot{z} z^{n-1}}{\sqrt{e+fz^n}}$

$$= \frac{2}{f} \dot{P}, \text{ and } \frac{\eta \dot{z} z^{n-1}}{g+bz^n \sqrt{e+fz^n}} = \frac{2\dot{P}}{fg - eb + bPP},$$

and

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and $\frac{d \dot{z} z^{\eta-1}}{g + bz^2 \times \sqrt{e + fz^2}} = \frac{2d}{\eta} \times \frac{\dot{P}}{fg - eb + bP^2}$, a
 Fluxion of the second Form, where $\theta = 0$, and
 where $d = \frac{2d}{\eta}$, $z = P$, $\eta = 2$, $e = fg - eb$, $f = b$,
 $R = \sqrt{\frac{eb - fg}{b}}$, $T = P$, or $\sqrt{e + fz^2}$, and
 $S = \sqrt{f \times \frac{g + bz^2}{b}}$: So that the Fluent sought is
 $\frac{2}{\eta fg - eb} dR \Big| \frac{R+T}{S}$, or $\frac{-2}{\eta eb - fg} dR \Big| \frac{R+T}{S}$.

PROBLEM XIV.

To compute any Number of Fluents at Pleasure of the ninth FORM.

Call any Fluent of the fifth *Form*, where $\theta = \lambda, c$, and call the two Fluents of the ninth *Form*, where $\theta = \lambda$ and $\lambda + 1$, a and b respectively, and we shall always have $ga + bb = c$: (See *Prob. XI.*) Whence if any one Fluent of the ninth *Form* be known, all the rest may be easily computed.

PROBLEM XV.

To compute any Number of Fluents at pleasure of the tenth FORM.

This is done by observing that every Fluent of the ninth *Form* where $\theta = \lambda$, by changing the Sign of η , the Quantities e and f for one another, as also the Quantities g and b for one another, is converted into a Fluent of the tenth *Form*, where

$\theta = 1 - \lambda$; and every Fluent of the ninth *Form*, where $\theta = -\lambda$, is converted into a Fluent of the tenth *Form*, where $\theta = 1 + \lambda$: For $\frac{d\dot{z} z^{\lambda-1}}{g + bz^n \times \sqrt{e + fz^n}}$

by this Change becomes $\frac{d\dot{z} z^{-\lambda-1}}{b + gz^{-n} \sqrt{f + ez^{-n}}}$.

Multiply $b + gz^{-n}$ into z^n , and also $\sqrt{f + ez^{-n}}$ into $\sqrt{z^n}$, and lastly multiply the Numerator into their Equivalent $z^{n+\frac{1}{2}}$, and the Fluxion will now

become $\frac{d\dot{z} z^{n-\lambda+\frac{1}{2}}}{g + bz^n \sqrt{e + fz^n}}$.

PROBLEM XVI.

To compute any Number of Fluents at pleasure of the seventh *Form*.

Call any Fluent of the seventh *Form*, where $\theta = \lambda$, c ; and call the two Fluents of the ninth *Form*, where $\theta = \lambda$ and $\lambda + 1$, a and b respectively; and we shall have $ea + fb = c$.

PROBLEM XVII.

To compute any Number of Fluents at pleasure of the eighth *Form*.

This is done by observing that every Fluent of the seventh *Form*, where $\theta = \lambda$, by changing the Sign of n , and the Quantities e and f for each other, as also the Quantities g and b for each other, is converted into a Fluent of the eighth *Form*, where $\theta = -\lambda$; as is easy to see.

L

PRO-

PROBLEM XVIII.

To find the Fluent of this Fluxion of the eleventh

$$\text{FORM } d\dot{z} z^{n-1} \times \sqrt{\frac{e+fz^n}{g+bz^n}}.$$

Make $\sqrt{g+bz^n} = Q$, and we shall have $z^n = \frac{Q^2 - g}{b}$, and $e+fz^n (=P) = \frac{eb-fg+fQ^2}{b}$: More-

over we shall have $\eta \dot{z} z^{n-1} = \frac{2}{b} \dot{Q} Q$, and conse-

$$\text{quently } d\dot{z} z^{n-1} \times \sqrt{\frac{e+fz^n}{g+bz^n}} = \frac{2d}{\eta b} \dot{Q} \times \sqrt{\frac{eb-fg+fQ^2}{b}}$$

a Fluxion of the fourth Form, where $\theta=0$, where

$$d = \frac{2d}{\eta b}, z = Q, \eta = 2, e = \frac{eb-fg}{b}, f = \frac{f}{b}, R = \sqrt{\frac{f}{b}},$$

$$T = \sqrt{\frac{eb-fg+fQ^2}{bQ^2}} = \sqrt{\frac{e+fz^n}{g+bz^n}}, \text{ and } S = \sqrt{\frac{eb-fg}{bg+bz^n}}.$$

So that the Fluent sought is $\frac{1}{\eta b} dPQ + \frac{eb-fg}{\eta fb}$
 $dR \left| \frac{R+T}{S} \right.$. Q. E. I.

PROBLEM XIX.

To investigate a GENERAL THEOREM for continuing upwards and downwards all the Series's of Fluents comprehended under the eleven first Forms; and an infinite Number of others under the same Character; but which is chiefly intended for the eleventh FORM.

The general Character of the Fluxions of the eleven first Forms is $d\dot{z} z^{n-1} \times e \sqrt[n]{fz^n} \times \sqrt[n]{g+bz^n}^{r-1}$, observing

observing that in the six first *Forms* $\psi - 1 = 0$, or $\psi = 1$. Let then $d\dot{z} z^{n-1} \times e + f z^{m-1} \times g + b z^{n-1} = a$,
 $d\dot{z} z^{n-1} \times e + f z^{m-1} \times g + b z^{n-1} = b$, and
 $d\dot{z} z^{n-1} \times e + f z^{m-1} \times g + b z^{n-1} = c$. Let
 moreover $z^n = N$, $e + f z^{m-1} = P$, and $g + b z^{n-1} = Q$,
 and from the Equation $z^n = N$, we shall have
 $\theta \dot{z} z^{n-1} = \dot{N}$, and $\theta d\dot{z} z^{n-1} \times e + f z^{m-1} \times g + b z^{n-1} =$

$\frac{d}{n} N \times e + f z^{m-1} \times g + b z^{n-1}$. Multiply

both Sides into $e + f z^{m-1} \times g + b z^{n-1}$, and we shall have

$\theta e g a + \theta \times e b + f g \times b + \theta f b c = \frac{d}{n} N P Q$, which I

call *the Fluxion of the first Part of the general Equation*.

Again, since $e + f z^{m-1} = P$ (by considering $e + f z^{m-1}$ as one single Quantity) we shall have

$\omega \dot{z} z^{n-1} \times e + f z^{m-1} = \dot{P}$, and $\omega \dot{z} z^{n-1} \times e + f z^{m-1} \times g + b z^{n-1} = N \dot{P} \times g + b z^{n-1}$, and

$\omega f d\dot{z} z^{n-1} \times e + f z^{m-1} \times g + b z^{n-1}$, or $\omega f b = \frac{d}{n}$

$N \dot{P} \times g + b z^{n-1}$. Multiply both Sides by $g + b z^{n-1}$,

and we shall have $\omega f g b + \omega f b c = \frac{d}{n} N \dot{P} Q$, which

is *the Fluxion of the second Part of the general Equation*.

Lastly, since $g + b z^{n-1} = Q$, we have

$\psi \dot{z} z^{n-1} \times g + b z^{n-1} = \dot{Q}$, and $\psi b d\dot{z} z^{n-1} \times$

$e + f z^{m-1} \times g + b z^{n-1}$, or $\psi b b = \frac{d}{n} N \times e + f z^{m-1} \times \dot{Q}$.

Multiply both Sides into $e + f z^{m-1}$, and we shall

have $\psi e b i + \psi f b c = \frac{d}{\eta} N P Q$; which is the Fluxion of the last Part of the general Equation. Put now these three fluxional Parts together, and make the Fluents equal, and we shall have

$$\theta e g a + \theta + \omega \times f g + \theta + \psi \times e b \times b + \theta + \omega + \psi \times f b c = \frac{d}{\eta} N P Q = \frac{d}{\eta} z^{\eta} \times \overline{e + f z^{\eta}}^{\omega} \times \overline{g + b z^{\eta}}^{\psi}. \quad Q. E. I.$$

PROBLEM XX.

To apply the foregoing Theorem to the Computation of Fluents of the eleventh FORM.

In order to do this it must be observed, that in this Form $\omega = \frac{1}{2}$, $\psi = \frac{1}{2}$, and that every Fluent of this Form where $\theta = \lambda$, by changing the Sign of η , the Quantities e and f , as also g and b for each other, is converted into a Fluent of the same Form, where $\theta = -\lambda$. This premised, let $d \dot{z} z^{-1} \times \sqrt{\frac{e + f z^{\eta}}{g + b z^{\eta}}} = a$, $d \dot{z} z^{-1} \times \sqrt{\frac{e + f z^{\eta}}{g + b z^{\eta}}} = b$, and $d \dot{z} z^{-1} \times \sqrt{\frac{e + f z^{\eta}}{g + b z^{\eta}}} = c$, and we shall have $\theta = -1$, and the general Equation applied here gives $-e g a + \frac{f g + e b}{2} \times b + f b c = \frac{d}{\eta} z^{-1} \times \overline{e + f z^{\eta}}^{\frac{1}{2}} \times \overline{g + b z^{\eta}}^{\frac{1}{2}} = \frac{e + f z^{\eta}}{\eta z^{\eta}} \times d P Q$, making $P = \sqrt{e + f z^{\eta}}$, as the Author has here defined it, and not equal to $\overline{e + f z^{\eta}}^{\frac{1}{2}}$, as above. But of these three Fluents, a , b , and c , c is

π is given by *Problem XVIII.* whence a will easily be found; because, from what has already been observed, c is convertible into a ; and these two a and c being obtained, the middle Fluent b will be had by the Equation above; after which all the rest will be easily computed. Q. E. F.

PROBLEM XXI.

To compute the Fluents of the twelfth FORM.

Make $\sqrt{g + bz^n} = Q$, and by the eighteenth Problem, we have $d\dot{z} z^{n-1} \times \sqrt{\frac{e + fz^n}{g + bz^n}} = \frac{2}{n} \frac{d\dot{Q}}{b} \times$

$$\sqrt{\frac{eb - fg + fQQ}{b}}; \text{ and consequently } \frac{d\dot{z} z^{n-1}}{k + lz^n} \times \sqrt{\frac{e + fz^n}{g + bz^n}} = \frac{2d}{n} \times \frac{\dot{Q}}{kb - lg + fQQ} \times \sqrt{\frac{eb - fg + fQQ}{b}},$$

a Fluxion of the eighth Form; where $\theta = Q$,

$$d = \frac{2d}{n}, z = Q, n = 2, e = \frac{eb - fg}{b}, f = \frac{f}{b}, g = kb - lg,$$

$$b = l, R = \sqrt{\frac{el - fk}{gl - bk}}, T = \sqrt{\frac{e + fz^n}{g + bz^n}}, \text{ and } S =$$

$$\sqrt{\frac{fg - eb \times k + lz^n}{gl - bk \times g + bz^n}}, r = \sqrt{\frac{f}{b}}, t = \sqrt{\frac{e + fz^n}{g + bz^n}}, s =$$

$$\sqrt{\frac{eb - fg}{g \times g + bz^n}}, \text{ and the Fluent is } \frac{-2}{nl} dR \left| \frac{R + T}{S} \right|$$

$$+ \frac{2}{nl} dr \left| \frac{r + t}{s} \right|; \text{ which Fluent being now known,}$$

the rest will be easily deduced from the eleventh Form, by the Method of the 14th Prob. Q. E. F.

PROBLEM XXII.

To find the Fluents of the two following Fluxions of the thirteenth FORM, viz. of $\frac{d\dot{z} z^{n-1}}{e+fz^n+gz^{2n}}=a$, and $\frac{d\dot{z} z^{2n-1}}{e+fz^n+gz^{2n}}=b$.

In order to destroy the middle Term of the Trinomial $e+fz^n+gz^{2n}$, and so to turn it into a Binomial, that the Fluxion may be reduced to some of the foregoing Problems: Let $\frac{1}{2}f+gz^n=T$, and we shall have $\frac{1}{4}ff+fz^n+gz^{2n}=T^2$, and $\frac{\frac{1}{4}ff+fz^n+gz^{2n}}{g}=\frac{T^2}{g}$, and $e+fz^n+gz^{2n}=\frac{e g - \frac{1}{4}ff + T T}{g}$. Put $R=\sqrt{\frac{1}{4}ff - e g}$, and we shall have $e+fz^n+gz^{2n}=\frac{T^2-R^2}{g}=\frac{S^2}{g}$, putting S^2 for T^2-R^2 . Further, since $\frac{1}{2}f+gz^n=T$, we have $ng\dot{z} z^{n-1}=\dot{T}$, and $\frac{d\dot{z} z^{n-1}}{e+fz^n+gz^{2n}}$, or $a=\frac{d}{n}\times\frac{\dot{T}}{S^2}=\frac{d}{nq}\times\frac{R^2\dot{T}}{S^2}$, putting q for R^2 , whose Fluent, by the second Problem, is $\frac{-1}{nq}dR\left|\frac{R+T}{S}\right.$.

Now to find b , make $e+fz^n+gz^{2n}=x$, and we shall have $nf\dot{z} z^{n-1}+2ng\dot{z} z^{2n-1}=\dot{x}$, and $\frac{nf d\dot{z} z^{n-1}+2ng d\dot{z} z^{2n-1}}{e+fz^n+gz^{2n}}$, or $nf a+2ng b=\frac{d\dot{x}}{x}$; therefore $nf a+2ng b=d\left|\frac{x}{e}\right.=d\left|\frac{e+fz^n+gz^{2n}}{e}\right.$.

Make

Make $N = 1 \left| \frac{e + f z^n + g z^{2n}}{e} \right|$, and we shall have
 $n f a + 2 n g b = d N$; by which Equation, since a is
 already found, b will easily be known.

PROBLEM XXIII.

To compute the Fluents of the thirteenth FORM.

Let $\frac{d \dot{z} z^{2n-1}}{e + f z^n + g z^{2n}} = a$, $\frac{d \dot{z} z^{2n+n-1}}{e + f z^n + g z^{2n}} = b$,
 $\frac{d \dot{z} z^{2n+2n-1}}{e + f z^n + g z^{2n}} = c$, and we shall have $ea + fb + gc =$
 $\frac{d}{\theta n} z^{2n}$; by Means of which Theorem, the Series
 of Fluents from those found in the last Problem
 may be continued upwards *ad infinitum*; and the
 rest may be found by considering that every
 Fluent of the thirteenth Form, where $\theta = \lambda$ is by
 changing the Sign of n , and the Quantities e and g
 for one another, convertible into a Fluent of the
 same Form, where $\theta = 2 - \lambda$.

N. B. Since in this Form, the Fluent where
 $\theta = 1$ is $\frac{1}{n q} d R \left| \frac{R + T}{S} \right|$, and, since this Fluent is
 by the usual Methods convertible into itself, we
 shall have $\frac{1}{n q} d R \left| \frac{R + T}{S} \right| = \frac{1}{n q} d r \left| \frac{r + t}{s} \right|$, and
 consequently $R \left| \frac{R + T}{S} \right| = r \left| \frac{r + t}{s} \right|$.

PROBLEM XXIV.

• To find the Fluents of the fourteenth Form.

Let $k+lz^n=y$, and we shall have $e+fz'+gz^{2n}$
 $= \frac{ell-fkl+gkk+fl-2gk \times y+gyy}{ll}$, and

$$\frac{d \dot{z} z^{n-1}}{k+lz^n+gz^{2n}} = \frac{\frac{d}{\eta} y}{y \times \frac{ell-fkl+gkk+fl-2gk \times y+gyy}{ll}}$$

a Fluxion of the thirteenth Form; where $\theta=0$,

$$d = \frac{d}{\eta}, z=y, \eta=1, e = \frac{ell-fkl-gkk}{l}, f = \frac{fl-2gk}{l},$$

$$R = \sqrt{\frac{1}{4}ff - eg}, T = \frac{1}{2}f + gz^n, S = \sqrt{g \times e + fz' + gz^{2n}},$$

$$q = \frac{1}{4}ff - eg, n=1 \left| \frac{ell-fkl+gkk+fl-2gk \times y+gyy}{gyy} \right|$$

$$= \frac{ll \times e + fz' + gz^{2n}}{g \times k + lz^{1/2}}. \text{ From the Measure of this}$$

Ratio subtract the Measure of the given Ratio of ll to g , which cannot affect the Fluent as such;

$$\text{and then we may make } n=1 \left| \frac{e+fz'+gz^{2n}}{k+lz^{1/2}} = M; \right|$$

and this Fluent being once found, the rest are easily derived from the thirteenth Form, according to the Manner of the fourteenth Problem.

N. B. Every Fluent of this fourteenth Form, where $\theta=\lambda$ is by changing the Sign of η , the Quantities e and g for one another, and k and l for one another, convertible into a Fluent of the same Form, where $\theta=3-\lambda$.

SCHO-

SCHOLIUM.

At the End of his eighteen *Forms*, the Author gives us other Values of R, T, and S, at least of T and S for every Form, which may be used at pleasure instead of those put down in the respective Forms: And before we can proceed any further, it will be necessary to shew whence, and how these Values are derived. Let then in any of the Forms

$R=a$, $T=b$, $S=c$, and we shall have $\frac{R+T}{S}$

$$= \frac{a+b}{c} = \frac{ab+aa}{ac} = \frac{a+\frac{aa}{b}}{\frac{ac}{b}}.$$

Make now $R=a$,

$T=\frac{aa}{b}$, $S=\frac{ac}{b}$, and we shall have $\frac{R+T}{S}$ in this

Case equal to $\frac{R+T}{S}$ in the former, and $R \left| \frac{R+T}{S} \right.$

in this Case equal to $R \left| \frac{R+T}{S} \right.$ in that.

But though the *logometrical* Fluents thus expressed agree entirely with those of the *Forms*, yet the *trigonometrical* Fluents deduced from them will be very different; for that a *logometrical* Expression may, when Occasion requires, be converted into a proper *trigonometrical* one, it is necessary that $T^2 - R^2$ should be equal to $+SS$, whereas according as R, T, and S are here defined, $T^2 - R^2$ is equal to $-S^2$; therefore to answer both the Ends of the *Forms*, we must correct the Value of S thus.

S thus. Instead of making $S = \frac{ac}{b}$, and $S^2 = \frac{a^2 c^2}{b^2}$,

let us make $SS = \frac{-a^2 c^2}{b^2}$, and $S = \sqrt{-\frac{a^2 c^2}{b^2}}$.

This done, the Fluent $R \left| \frac{R+T}{S} \right|$, when $S = \frac{ac}{b}$,

and the Fluent $R \left| \frac{R+T}{S} \right|$ when $S = \sqrt{-\frac{a^2 c^2}{b^2}}$ will

both be Fluents of the same Fluxion, as differing from one another only by a constant Quantity, viz. by the Measure of the Ratio of 1 to $\sqrt{-1}$; and therefore if the *logometrical* Part was true before this Correction was made, it will be so still; and as $TT - RR$ is now equal to $+SS$, it is plain that the *logometrical* Expression will now be convertible into a proper *trigonometrical* one.

PROBLEM XXV.

To find the Fluent of this Fluxion of the sixteenth

$$\text{FORM } \frac{d \dot{z} z^{n-1}}{\sqrt{e + f z^n + g z^{2n}}}.$$

Make $\frac{1}{2} \frac{f}{g} + z^n = y$, and we shall have $\frac{1}{4} \frac{ff}{g} + \frac{fz^n}{g} + z^{2n} = yy$, and $\frac{1}{4} \frac{ff}{g} + fz^n + g z^{2n} = gyy$, and $e + fz^n + g z^{2n} = \frac{eg - \frac{1}{4}ff}{g} + gyy$. We have moreover, from the first Equation, $\eta \dot{z} z^{n-1} = \dot{y}$, and there-

therefore $\frac{d \dot{z} z^{\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}} = \frac{\frac{d}{\eta} \dot{y}}{\sqrt{e+g-\frac{1}{4}ff} + gyy}$

a Fluxion of the sixth Form, where $\theta=0$, $d=\frac{d}{\eta}$,

$z=y$, $\eta=2$, $e=\frac{eg-\frac{1}{4}ff}{g}$, $f=g$, $R=\sqrt{g}$, $T=g \times$

$\frac{\sqrt{e+fz^{\eta}+gz^{2\eta}}}{\frac{1}{2}f+gz^{\eta}}$, $S=\frac{\sqrt{g \times eg-\frac{1}{4}ff}}{\frac{1}{2}f+gz^{\eta}}$. These are

the Values of R, T, and S as they immediately flow from our Notation; but if we take the Values of R, T, and S according to the last *Scholium*

we shall have $R=\sqrt{g}$, $T=\frac{\frac{1}{2}f+gz^{\eta}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}$,

$S=\sqrt{\frac{\frac{1}{4}ff-eg}{e+fz^{\eta}+gz^{2\eta}}}$: And these Values (for their

greater Simplicity) the Author chuses rather than the other, and the Fluent of this Fluxion

$\frac{d \dot{z} z^{\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}$ will be $\frac{1}{\eta g} dR \left| \frac{R+T}{S} \right.$.

PROBLEM XXVI.

To continue the Series of Fluents of the sixteenth FORM.

Let $\frac{d \dot{z} z^{\theta\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}} = a$, $\frac{d \dot{z} z^{\theta\eta+\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}} = b$,

$\frac{d \dot{z} z^{\theta+2\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}} = c$, $z^{\theta\eta} = N$, and $\sqrt{e+fz^{\eta}+gz^{2\eta}} = P$,

and

and we shall have $\theta d\dot{z} z^{\theta\eta-1} = \frac{d}{\eta} \dot{N}$, and $\theta d\dot{z} z^{\theta\eta-1}$
 $\times \sqrt{e+fz^{\eta}+gz^{2\eta}}$, or $\theta d\dot{z} z^{\theta\eta-1} \times \frac{e+fz^{\eta}+gz^{2\eta}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}$,
 or $\theta e\dot{a} + \theta f\dot{b} + \theta g\dot{c} = \frac{d}{\eta} \dot{N} P$; which will be the
Fluxion of the first Part of the general Equation.
 Again, since $e+fz^{\eta}+gz^{2\eta} = PP$, we have $\frac{1}{2}\eta f\dot{z} z^{\eta-1}$
 $+ \eta g\dot{z} z^{2\eta-1} = \dot{P}P$, and $\frac{\frac{1}{2}\eta f\dot{z} z^{\eta-1} + \eta g\dot{z} z^{2\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}} = \frac{d}{\eta} \dot{P}$,
 and $\frac{\frac{1}{2}\eta f\dot{z} z^{\theta\eta-1} + \eta g\dot{z} z^{\theta\eta+2\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}$, or $\frac{1}{2}f\dot{b} + g\dot{c} = \frac{d}{\eta} \dot{N} \dot{P}$;
 which is the *Fluxion of the second Part of the general Equation.* Join these two Parts together, and make the Fluents equal, and there will arise this general Equation $\theta e a + \theta + \frac{1}{2} \times f b + \theta + 1 \times g c = \frac{d}{\eta} \dot{N} P = \frac{z^{\theta\eta}}{\eta} dP$; which Equation, by the Help of the twenty fifth *Problem*, will find any Number of Fluents of this Form proposed.

N. B. Every Fluent of this *Form*, where $\theta = \lambda$ is by changing the Sign of η , and the Quantities e and g for one another, convertible into a Fluent of the same *Form*, where $\theta = 1 - \lambda$.

PROBLEM XXVII.

To find the Fluents of the fifteenth Form.

The Fluents of this Form are derived from those of the sixteenth in the same Manner as the Fluents of the third and seventh Forms are derived from those of the fifth and ninth.

PRO-

PROBLEM XXVIII.

To find the Fluents of the eighteenth FORM.

Let $k + lz^n = y$, and we shall have $e + fz^n + gz^{2n}$

$$= \frac{ell - fkl + gkk + fl - 2gk \times y + gyy}{ll \quad \frac{d}{\eta}}, \text{ and}$$

$$\frac{d \dot{z} z^{n-1}}{k + lz^n \times \sqrt{e + fz^n + gz^{2n}}} = \frac{y \times \sqrt{ell - fkl + gkk + fl - 2gky + gyy}}{\frac{d}{\eta}},$$

a Fluxion of the sixteenth Form, where $\theta = 0$, $d = \frac{d}{\eta}$

$z = y$, $\eta = 1$, $e = ell - fkl + gkk$, or p , $f = fl - 2gk$,

and $g = g$; $r = \sqrt{p}$, $t = \frac{l \times e + \frac{1}{2} f z^n - k \times \frac{1}{2} f + g z^n}{\sqrt{e + fz^n + gz^{2n}}}$,

$s = k + lz^n \times \sqrt{\frac{1}{4} ff - eg}$; and the Fluent itself

equals $\frac{-1}{\eta p} dr \left| \frac{r+t}{s} \right.$. But if, as the Author has

done, we ascribe the aforesaid Values to R, T, and

S, the Fluent will then be $\frac{-1}{\eta p} dR \frac{R+T}{S}$; and

this Fluent being obtained all the rest are deducible from the sixteenth Form; after the same Manner as the seventh was deduced from the ninth.

N. B. Every Fluent of this Form where $\theta = \lambda$, by changing the Sign of η , and the Quantities e and g for one another, and k and l for one another, becomes a Fluent of the same Form, where $\theta = 2 - \lambda$.

PRO-

PROBLEM XXIX.

To find the Fluents of the seventeenth FORM.

This Form is derived from the eighteenth, just in the same Manner as the third is derived from the fifth, as is easy to see; and every Fluent of this Form, where $\theta = \lambda$ is convertible into a Fluent of the same Form, where $\theta = -\lambda$.

Of the PREPARATION of TABULAR FLUENTS.

(Translated from a Latin Manuscript of the Author.)

The Fluents of fluxionary Expressions, as they stand in the Tables, do not always come out in their simplest Form; but require, before they can be of Use, to be some how prepared or transformed; either by adding or subtracting some given Quantity; by increasing or diminishing the *Modulus*; by exterminating an *impossible* Quantity; or, perhaps, by reducing the *Tabular Ratio* to more simple Terms. I have thought proper to give an Instance or two, from Mr. Cotes's Tables, wherein a Difficulty of this kind occurs.

EXAMPLE I.

Let the Fluent proposed be that of the Fluxion

$$\frac{2 a a \dot{x}}{a a - x x};$$

in which x and a represent right Lines, the former *variable*, and the other *invariable*; and a is likewise supposed to be greater than x .

2

Now,

of TABULAR FLUENTS. 159

Now, as the Denominator of this Fluxion is *rational*, it must belong to the *first*, or to the *second* Form, according as the Number θ is a *whole* Number, or a *Fraction*. But, in our Example, seeing $\eta=2$, and $\theta\eta=1$, or $2\theta-1=0$, we have $\theta=\frac{1}{2}$, and *Form* II. is that which we are to use. In which, $d=2aa$, $z=x$, $\eta=2$, $\theta=0$, $e=aa$, $f=-1$, $R=a$, $T=x$, $S=\sqrt{aa-xx}$; and

$$\frac{2}{\eta e} dR \left| \frac{R+T}{S} = 2a \left| \frac{a+x}{\sqrt{aa-xx}} \right. \right.$$

And now, to exterminate the impossible Quantity in the Denominator S ; to the Ratio $\frac{a+x}{\sqrt{aa-xx}}$

I add the Ratio $\frac{1}{\sqrt{-1}}$ (whose *Measure*, whatever it be, is at least determinate and invariable) and the Sum is the Ratio $\frac{a+x}{\sqrt{aa-xx}}$, or $\frac{\sqrt{aa+2ax+xx}}{\sqrt{aa-xx}}$;

whence the Fluent sought is $2a \left| \frac{\sqrt{aa+2ax+xx}}{\sqrt{aa-xx}} \right.$, or $a \left| \frac{(a+x)^2}{aa-xx} \right.$: For the Measure of a Ratio to any

Modulus is equivalent to the Measure of the Duplicate of the same Ratio to half that *Modulus*.

But the Ratio $\frac{a^2+2ax+x^2}{a^2-xx}$, dividing both Terms by $a+x$, becomes $\frac{a+x}{a-x}$: And the Fluent thus prepared

prepared is $a \left| \frac{a+x}{a-x} \right|$; or the Measure of the Ratio of $a+x$ to $a-x$ to the *Modulus* a .

EXAMPLE II.

Let it now be proposed to find the Fluent of $\frac{ax}{\sqrt{a^2-x^2}}$ by *Form* VI. where we have $d=a$, $z=x$, $n=2$, $\theta=0$, $e=aa$, $f=-1$, $R=\sqrt{-1}$, $T=\frac{\sqrt{a^2-x^2}}{x}$, $S=\frac{a}{x}$. And because, according to this Notation, the Quantity R , and thence the *Modulus* $\frac{2}{nf}dR$, comes out *impossible*, we infer that the Fluent sought is not the *Measure* of a *Ratio*, but of an *Angle**: And that the *Modulus* of the *trigonometrical* Measure, is the *Modulus* of the *logometrical* Measure made *possible*.

I change therefore the Sign of -1 in the Value of R , putting $R=\sqrt{+1}$, or 1 ; and thence

$$\frac{2}{nf}dR \left| \frac{R+T}{S} \right| = -a \left| \frac{1 + \frac{\sqrt{a^2-x^2}}{x}}{\frac{a}{x}} \right|, \text{ or } -a \left| \frac{x + \sqrt{a^2-x^2}}{a} \right|;$$

and from this Notation conclude, that the Fluent sought is the Measure of an *Angle* to the *Modulus* $-a$, the *Radius*, *Tangent*, and *Secant* of that *Angle* being as x , $\sqrt{a^2-x^2}$ and a ; or where the *Radius* is to the *Secant* as x to a : for of the three Terms,

* See *Logometria*, Page 45.

Radius,

Radius, Tangent, and Secant, any two determine the Magnitude of an Angle. Our Fluent therefore is the *affirmative* Measure of an Angle whose Radius is to its Secant as x to a , the absolute Magnitude of the Radius being $-a$, or it is the *negative* Measure of the same Angle to the Radius $+a$: for an *affirmative* Measure (whether *logometrical* or *trigonometrical*) to a negative *Modulus*, is the same Thing as a *negative* Measure to an affirmative *Modulus*.

Now to construct this Fluent in its simplest Form, in Fig. 27. Plate V. from the Center C, at the Distance $CA=a$ describe a Quadrant AMB ; and in the Radius CA having taken $CP=x$, and erected the Perpendicular PM , the Arc $-AM$ will be the Fluent sought. For joining CM , the Radius of the Angle ACM will be to the Secant as x to a : To this Fluent add the Quadrantal Arc AMB , and the Arc $-AM$ will be changed into BM . And therefore the Fluent of the Fluxion proposed is a circular Arc whose Radius is a and its right Sine x .

ANALYSIS of the PROBLEMS in the
SCHOLIUM GENERALE of Mr.
Cotes's LOGOMETRIA *.

PROBLEM I. Fig. 28.

To find the Length of the parabolic Curve AP.

ANALYSIS.

Call the *Parameter*, a ; the *Abscisse* A Q, x ; and the *Ordinate* P Q, y ; and we shall have $a x = y^2$, and $a \dot{x} = 2 y \dot{y}$, and $\dot{x} = \frac{2 y \dot{y}}{a}$, and $\sqrt{x x + y y}$, or the Fluxion of the Curve, $= \frac{\dot{y}}{a} x \sqrt{a a + 4 y y}$, a Fluxion of the *fourth Form*; where $\theta = 0$, and $\frac{x^n}{n} dP$, or the *rational Part* of the Fluents, $= \frac{1}{4} y x \frac{\sqrt{a a + 4 y y}}{a}$; and $\frac{e}{nf} dR \left| \frac{R+T}{S} \right.$, or the *irrational Part*, equal to the Measure of the Ratio, *ad Modulum* $\frac{1}{4} a$, between $2 + \frac{\sqrt{a^2 + 4 y^2}}{y}$ and $\frac{a}{y}$.

Now to construct this Fluents, let PT be a Tangent to the *Parabola* in the Point P, cutting QA produced in T; and since QT $= 2 x$, or $\frac{2 y y}{a}$, the Tangent PT will be $y x \frac{\sqrt{a a + 4 y y}}{a}$; whence the *rational Part* of our Fluents $\frac{1}{4} y x \frac{\sqrt{a a + 4 y y}}{a}$ will be

* *Harmonia Mensurarum*, Pag. 22.

equal

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equal to $\frac{1}{2}$ PT or AL*. Now as to the *irrational* Part, which is the Measure of the Ratio, *ad Modulum*

$\frac{1}{4}a$, between $2 + \frac{\sqrt{aa+yy}}{y}$ and $\frac{a}{y}$, or between $\frac{yy}{a}$

$+ \frac{1}{2}y \times \frac{\sqrt{aa+4yy}}{a}$ and $\frac{y}{2}$; it is evident that the

Modulus $\frac{1}{4}a = AF$, F being the *Focus*; that $\frac{yy}{a}$

or $x = AQ$, that $\frac{1}{2}y \times \frac{\sqrt{aa+4yy}}{a} = AL$; and

lastly that $\frac{1}{2}y$ is LQ. So that if to the Line AL be added the Line LM equal to the Measure of the Ratio between $AQ + AL$ and LQ, in a System whose *Modulus* is AF, the whole Line AM will be the Length of the Curve AP required. Q. E. I.

PROBLEM II. Fig. 29.

To measure any Part QP of the Archimedean SPIRAL.

ANALYSIS.

Draw the indefinite Line QZ representing the Position of the *Radius primus*, and the Nature of this *Spiral* is, that the Length of any other Radius QP must always be as the Angle PQZ. This premised, draw the Radius Qp greater than, and infinitely near QP; and setting off upon Qp, QU equal to QP, and joining PU; since the Angle PQZ is always as QP, the Angle PQp will always be as Up; and consequently PU which will ever be as the Angle PQp, and the Radius PQ together, must always be as

*The Construction being AL drawn parallel to TP, or QL=LP.

M 2

QP × UP;

$QP \times Up$; and consequently $QP \times \frac{Up}{UP}$ will be a constant Quantity. Call this Quantity $\frac{1}{2}a$, and QP, y , and $UP, \dot{x}$; and we shall have $\dot{x} = \frac{2y\dot{y}}{a}$, and $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, or the Fluxion of the Curve $= \frac{\dot{y}}{a} \times \sqrt{a^2 + 4y^2}$ as in Prob. I. and therefore the Fluent will be $\frac{1}{2}y \times \frac{\sqrt{a^2 + 4y^2}}{a}$, together with the Measure of the Ratio between $\frac{yy}{a} + \frac{1}{2}y \times \frac{\sqrt{a^2 + 4y^2}}{a}$ and $\frac{1}{2}y$, the *Modulus* being $\frac{1}{4}a$.

To construct which, let PT be a Tangent to the Spiral in the Point P , cutting a Line as QT perpendicular to QP in T , and the similar Triangles UpP, QPT give us $Up (\dot{y})$ to $Pp (\frac{\dot{y}}{a} \times \sqrt{a^2 + 4y^2})$ as $QP (y)$ is to $PT = \frac{y}{a} \times \sqrt{a^2 + 4y^2}$; and therefore $\frac{1}{2}y \times \frac{\sqrt{a^2 + 4y^2}}{a} = \frac{1}{2}PT$ or AL^* .

Again, since $\frac{1}{2}a = \frac{QP \times Up}{UP}$, or $\frac{QP \times QP}{QT}$, or $\frac{QP \times QL}{QA}$; $\frac{1}{4}a$, or the *Modulus*, will be equal to $\frac{QL^2}{QA}$, or a third Proportional to the Lines AQ and QL ; and therefore will be equal, by the Construction, to AF . Lastly, for the Terms of the

* AL parallel to PT bisects QP in L , and AF is a third Proportional to QA, QL , by Construction.

Ratio, since $\frac{1}{4}a = \frac{\overline{QL}^2}{AQ}$, and consequently $a = \frac{\overline{QP}^2}{AQ}$,

we shall have $AQ = \frac{\overline{QP}^2}{a}$, or $\frac{yy}{a}$, and the rest are

already defined. So that if to the Line AL you add the Line LM equal to the Measure of the Ratio between $AL + AQ$ and LQ *ad Modulum* AF, the whole Line AM will be the Length of the Curve QP required. Q. E. I.

N. B. This Affinity between the *Spiral* in this Problem, and the *Parabola* in the last, arises from hence, that the latter is nothing else but the *Involute* of the former.

PROBLEM III. Fig. 30.

To find the Length of any Part as Ee of the Reciprocal SPIRAL.

ANALYSIS.

Draw the Radius Aε greater than AE, but infinitely near it; and upon the Center A, and with the Radius AE describe the Arc EGH cutting Aε in G, and AB in H; and that Arc EH will be to the Arc GH as the Angle EAH is to the Angle GAH; that is, (from the Nature of the *Spiral*) as Aε to AE; and by Division EH is to EG as Aε or AE is to Gε; and by Alternation EH is to AE as EG is to Gε, or as AF is to AE; and therefore the Subtangent AF will always be equal to the Arc EH: But that

M 3

Arc

Arc being as the Radius A E, and the Angle E A B together, will be a constant Quantity, since the Angle E A B is reciprocally as the same Radius A E; and therefore the Subtangent A F must be so too.

This premised, call A F, a ; A E, y ; and E G, $\dot{x}$; and we shall have $\dot{x} = \frac{a\dot{y}}{y}$, and $\sqrt{\dot{x}^2 + \dot{y}^2}$, or the Fluxion of the Curve $= \frac{\dot{y}}{y} \times \sqrt{aa + yy}$; a Fluxion of the *third Form*, where $\theta = 0$, and whose Fluent $\frac{2}{y} dP - \frac{2}{y} dR \left| \frac{R + T}{S} = \sqrt{aa + yy} - a \right| \frac{a + \sqrt{aa + yy}}{y}$, whereof $\sqrt{aa + yy}$ is the Tangent E F, and $a \left| \frac{a + \sqrt{aa + yy}}{y} \right|$ is the Measure of the Ratio *ad Modulum* A F, between E F + F A and E A, or between E A and E F - F A = (by Construction) L M: For since $\overline{EA}^2 = \overline{EF}^2 - \overline{FA}^2 = \overline{EF + FA} \times \overline{EF - FA}$, it follows, that E F + F A is to E A, as E A is to E F - F A: So that the Fluent at E is E F - L M; and for the same Reason, the Fluent at e is $ef - lm$, and if this latter Fluent is subtracted from the former, the Difference E F - ef + lm - L M will be the Length of the Curve E e required. Q. E. I.

Plate V.

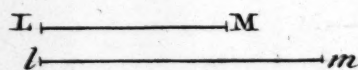
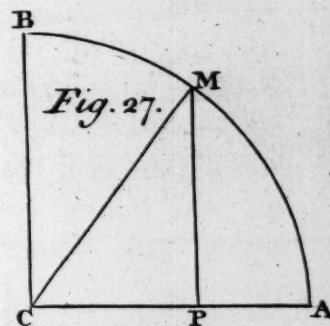
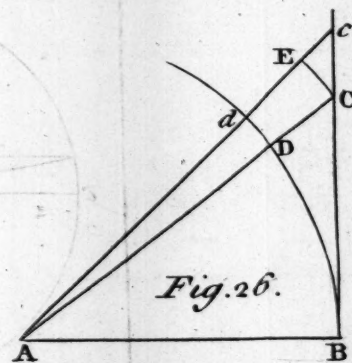
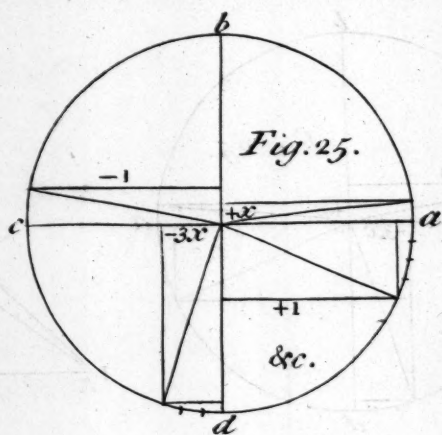


Fig. 28.

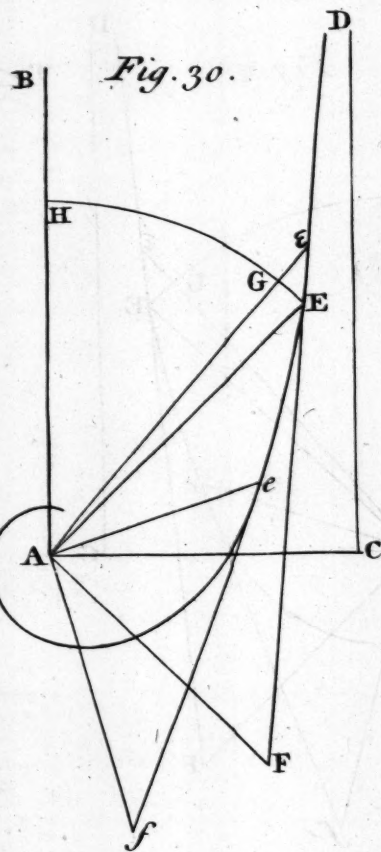
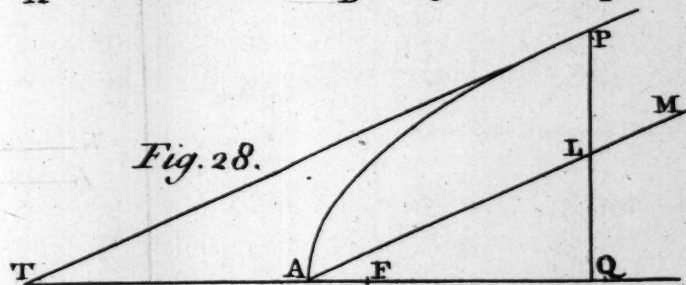
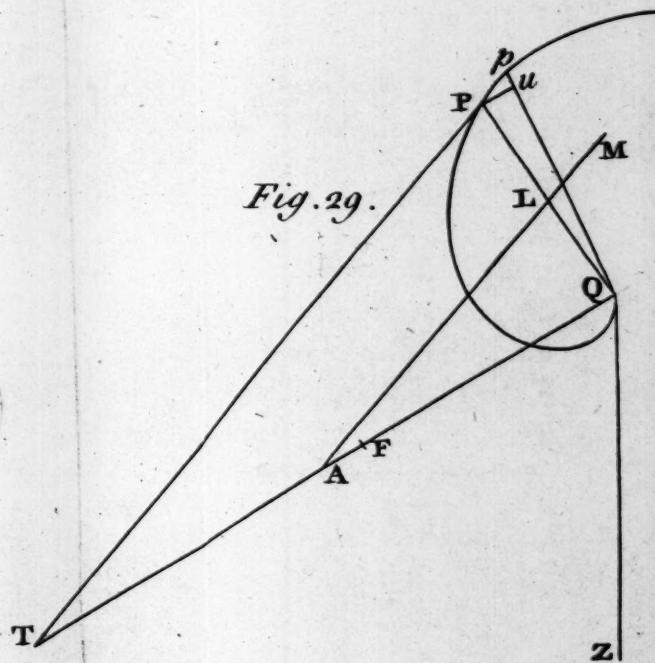


Fig. 30.

Fig. 29.



PROBLEM IV. Fig. 31.

To find the Length of any Part as Ee of the Logarithmic CURVE.

ANALYSIS.

Call the Subtangent AF , which is a constant Quantity, a ; call also Aa , x ; and AE , y ; and we shall have $\dot{x} = \frac{ay}{y}$, as before; and consequently the Length of the Curve Ee will be $EF - ef + lm - LM$, as before. Q. E. I.

Only here it must be observed, that to find the Lines LM and lm , there is no Necessity of having Recourse to the *Canon*, since this Curve, of its own Nature, affords us a perfect *System of Measures*: For if LM be taken equal to $EF - FA$, LM will be the Measure of the Ratio of AE to AL , or of AE to $EF - FA$ *ad Modulum* AF , from the very Nature of the Curve; and the same may be said of lm , which is the Measure of the Ratio of ae to al or $ef - fa$ *ad Modulum* af or AF .

N. B. This Curve is the *Evolute* of the Spiral in the last Problem.

PROBLEM V.

To find the Fluxion of this Fluent $e + fz^n$.

SOLUTION.

Make $e + fz^n = y$, and we shall have $\overline{e + fz^n}^w = y^w$, and the Fluxion of $\overline{e + fz^n}^w = wy^{w-1} = w\dot{y} \times \overline{e + fz^n}^{w-1}$.

M 4

But

found will form a Curve, which is called a *Cissoïd*; and the Circle whose Diameter is A B is called the *generating Circle*.

COROLL. I.

If to the End B of the Diameter A B is erected a Perpendicular as B C, that Line B C will be an *Asymptote* to the *Cissoïd*.

COROLL. II.

If we call the Diameter A B, a ; B Q, x ; and P Q, y ; the Nature of the *Cissoïd* will be expressed by this Equation $\frac{a-x^{\frac{1}{2}}}{x^{\frac{1}{2}}} = y$. But if A Q is called x , the Nature of the Curve will be $y = \frac{x^{\frac{1}{2}}}{a-x^{\frac{1}{2}}}$.

PROBLEM VII.

To find the Length of any Part, as A P, of the *CISSOID*.

ANALYSIS.

Call A B, a ; B Q, x ; and P Q, y ; and since, according to the Nature of the *Cissoïd*, $y = \frac{a-x^{\frac{1}{2}}}{x^{\frac{1}{2}}}$, or $x^{-\frac{1}{2}} \times \overline{a-x^{\frac{1}{2}}}$, we shall have, by the sixth *Problem*, $\dot{y} = \frac{-\frac{1}{2}\dot{x}}{x^{\frac{3}{2}}} \times \overline{a+2x} \times \sqrt{a-x}$, and $y^2 = \frac{\frac{1}{4}\dot{x}^2}{x^3}$ $\times \overline{a^3+3aax-4x^3}$: But $\dot{x}^2 = \frac{\frac{1}{4}\dot{x}^2}{x^3} \times 4x^3$; therefore $\dot{x}^2 + y^2 = \frac{\frac{1}{4}\dot{x}^2}{x^3} \times \overline{a^3+3aax}$, or $\frac{1}{4} \frac{a^2\dot{x}^2}{x^3} \times \overline{a+3x}$; and

and therefore $\sqrt{\dot{x}^2 + \dot{y}^2}$, or the Fluxion of the Curve, is $\frac{-\frac{1}{2}a\dot{x}}{x^{\frac{3}{2}}} \times \sqrt{a+3x}$; the Sign — being prefixed, because the Part AP of the Curve now inquired into flows contrarily to BQ or x , whose Fluxion is $+\dot{x}$.

But the foregoing Fluxion $\frac{-\frac{1}{2}a\dot{x}}{x^{\frac{3}{2}}} \times \sqrt{a+3x}$ is a Fluxion of the *fourth Form*, where $\theta = -1$, and whose Fluent consists of two Parts; the *rational* Part is $a \times \sqrt{\frac{a+3x}{x}}$ or $\frac{a}{x} \times \sqrt{ax+3xx}$, or $\frac{a}{x} \times \sqrt{ax-xx+4xx}$, or $\frac{2a}{x} \times \sqrt{\frac{ax-xx}{4}} + xx$.

To construct which, suppose the Ordinate QP to cut the generating Circle in O, and bisect both QP in F, and QO in G, and draw the Lines AFE, BG: This done $\frac{ax-xx}{4}$ will be equal to $\overline{QG}^2$, and xx to $\overline{QB}^2$; and therefore $\sqrt{\frac{ax-xx}{4}} + xx$ will be equal to the Line BG, and we shall have $\frac{2a}{x} \times \sqrt{\frac{ax-xx}{4}} + xx = \frac{2a}{x} \times BG$. But now since, according to the Nature of the *Cissoïd*, QP is to QA as QA to QO, and since from the Nature of the Circle QA is to QO as QO to QB, it follows, *ex aequo*, that QP is to QA as QO to QB; and bisecting the Antecedents, that QF is to QA as QG to QB; and therefore the Triangle BQG is

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is similar to the Triangle A Q F, and consequently also to the Triangle A B E; wherefore as B Q (x) is to B G, so is A B (a) to A E = $\frac{a}{x} \times$ B G: So that the *rational* Part of our Fluent is at last found to be 2 A E.

In the *irrational* Part R is to S as $\sqrt{3}$ is to $\sqrt{\frac{a}{x}}$, or as $\sqrt{3}x$ is to $\sqrt{a}$, or as $\sqrt{3ax}$ is to a , or as $\sqrt{ax}$ is to $\frac{a}{\sqrt{3}}$: But $\sqrt{ax}$ is a mean Proportional between a and x , or between B A and B Q, = B D*; and $\frac{a}{\sqrt{3}}$ is the Tangent of an Angle of thirty Degrees *ad Radium* $a =$ B C; and therefore R is to S as B D to B C, and consequently R + T is to S as B D + D C is to B C: For in all *logometrical* Fluents whatever, T is the Hypothenufe of a right-angled Triangle whose other two Sides are R and S, as will be evident from the Construction of the *Forms* themselves: And since the *Modulus* of this Part is $-\sqrt{3} \times a$, or $-\frac{3a}{\sqrt{3}}$, or -3 B C, the irrational Part will be -3 B C $\left| \frac{B D + D C}{B C} \right|$; and the whole Fluent will be 2 A E $- 3$ B C $\left| \frac{B D + D C}{B C} \right|$.

* By Construction.

Now

Now to determine exactly by the Help of this Fluent the Length of the Curve A P, it must be observed, That when the Point Q, and consequently P coincides with the Point A, the Line A E will become A B, and 2 A E, 2 A B: Subtract therefore 2 A B from the *rational* Part 2 A E, and the rest 2 A E—2 A B will be for our Purpose. In like manner it must be observed, That when the same Point Q, and consequently D, coincides

with A, $-3BC \left| \frac{BD+DC}{BC} \right|$ will become $-3BC \left| \frac{BA+AC}{BC} \right|$; and this latter subtracted from the

former leaves $+3BC \left| \frac{BA+AC}{BC} \right| - 3BC \left| \frac{BD+C}{BC} \right|$

also for our Purpose. But the Measure of the Ratio of BD+DC to BC being subtracted from the Measure of the Ratio of BA+AC to BC, in the same *System*, will leave the Measure of the Ratio of BA+AC to BD+DC. So that the Length of the Curve A P is at last found equal

to $2AE - 2AB + 3BC \left| \frac{BA+AC}{BD+DC} \right|$. Q. E. I.

PROBLEM VIII.

To find the Altitude of a Cylinder equal to, and upon the same Base with the Solid generated by the Revolution of the cissoïdal Area APQ about its Axe A Q.

ANALYSIS.

Call AB, a ; A Q, x ; and P Q, y ; and we shall have, from the Nature of the *Cissoïd*, $y = \frac{x\sqrt{x}}{\sqrt{a-x}}$, and $yy = \frac{x^3}{a-x}$; and if we suppose the Diameter of a Circle to be to the Circumference as 1 to d , we shall have $dyy\dot{x}$, or the Fluxion of the Solid, $= \frac{d\dot{x}x^3}{a-x}$, a Fluxion of the *first Form*; where $\theta=4$, and whose Fluent is $-\frac{dx^3}{3} - \frac{dax^2}{2} - daax - da^3 \left| \frac{a-x}{a} \right.$.

But the *irrational* Part may be expressed thus, $+da^3 \left| \frac{a}{a-x} \right.$; since subtracting the Measure of a negative Ratio is the same as adding the Measure of a positive one; and as this Fluent when the Point Q coincides with A is apparently equal to nothing, it will be a true and adequate Representative of the Content of the Solid now inquired into.

Put now z for the Altitude of a Cylinder equal to this Solid, and upon the same Base with it;
and

and we shall have $dy y z = -\frac{dx^3}{3} - \frac{da x^2}{2} - da^2 x - da^3 \left| \frac{a}{a-x} \right|$; whence z will be found by applying the several Members of the foregoing Series to the Quantity $dy y$, or $\frac{dx^3}{a-x}$; thus $\frac{dx^3}{3 dy y} = \frac{a-x}{3}$, $= \frac{BQ}{3}$, and $\frac{da x x}{2 dy y} = \frac{\frac{1}{2}a^2}{x} - \frac{1}{2}a = \frac{AR-AB}{2} = \frac{BP}{2}$, and $\frac{da^2 x}{dy y} = \frac{a^3}{x^2} - \frac{a^2}{x} = AS-AR=RS^*$. So that the *rational* Part of the Altitude is equal to $-\frac{QB}{3} - \frac{BR}{2} - RS$, equal to the Right Line $-XZ$. Again, $\frac{da^3}{dy^2} = \frac{a^4}{x^3} - \frac{a^3}{x^2} = AT-AS=ST$; and therefore the *irrational* Part of the Altitude will be $ST \left| \frac{a}{a-x} \right|$, (since whatever Change is made in the *Modulus* produces a like Change in the Measure) or $ST \left| \frac{AB}{BQ} \right|$ equal to the Right Line QX : So that the Altitude sought will be $QX - XZ = QZ$ †. Q. E. I.

* By Construction, AQ, AB, AR, AS, AT , are $\div$; $QX=TS \left| \frac{AB}{BQ} \right|$; and $XZ=SR+\frac{1}{2}RB+\frac{1}{3}BQ$.

† See *Logometr.* p. 24.

PROBLEM IX. Fig. 33.

To measure the conchoidal SOLIDS generated by the Revolution of the two Areas $AEDC$, $aeDC$ about the Axe AP .

ANALYSIS.

Draw the Line $PTbKH$ greater than PE , but infinitely near it, cutting the conjugate Curves AE and ae produced in H and b respectively, the Line CD in K , and the Circle RS (described from the Pole P at the Distance $PS = CA = Ca$) in T . Draw also SQ cutting PR at right Angles in Q , and call PC , a ; PR , r ; PQ , x ; SQ , y ; the Area of the whole Circle RST , b ; the Solid generated by the Area $AEEa$; or the sum of the Solids generated by the two Areas $AEDC$, $aeDC$, s ; the Difference of those Solids, d ; and the Solid generated by the Sector PRS , u ; and we shall have, *first*, The Circumference of the whole Circle $RST = \frac{2b}{r}$, and consequently the Circumference of the Circle whose Radius is SQ equal to $\frac{2by}{rr}$; and since ST , or the Fluxion of the Arc RS is equal to $\frac{-r\dot{x}}{y}$, we shall have the *Annulus* generated by the Line ST equal to $\frac{-2b\dot{x}}{r}$. But u , or the Solid generated by the nascent Area PST is equal to a Cone whose Base is the above-mentioned *Annulus*, and whose Altitude is PS ; where-

wherefore $\dot{u} = \frac{-2bx}{3}$, and $u = \frac{2br - 2bx}{3} = \frac{2b}{3}$

$\times r - x$, the Fluent at first being $-\frac{2br}{3}$. Again,

since the Areas PST , $Pe b$, and PEH are the same as to Magnitude as if they were similar, the Solids generated by them will be as the Cubes of the Lines PS , Pe , and PE respectively; and $\dot{u}$, or the Solid generated by the Area $EHbe$ will be to $\dot{u}$ as $\overline{PE^3} - \overline{Pe^3}$ to $\overline{PS^3}$: But $\overline{PE^3} = \overline{PD^3} + 3\overline{PD^2} \times DE + 3\overline{PD} \times \overline{DE^2} + \overline{DE^3}$, and $\overline{Pe^3} = \overline{PD^3} - 3\overline{PD^2} \times DE + 3\overline{PD} \times \overline{DE^2} - \overline{DE^3}$, and $\overline{PE^3} - \overline{Pe^3} = 6\overline{PD^2} \times DE + 2\overline{DE^3}$; wherefore $\dot{u}$ is to $\dot{u}$ as $6\overline{PD^2} \times DE + 2\overline{DE^3}$ to $\overline{PS^3}$, and $\frac{1}{2}\dot{u}$ is to $\dot{u}$ as $3\overline{PD^2} \times DE + \overline{DE^3}$ is to $\overline{PS^3}$, and by Division $\frac{1}{2}\dot{u} - \dot{u}$ is to $\dot{u}$ as $3\overline{PD^2} \times DE$ is to $\overline{PS^3}$, or as $3\overline{PD^2}$ is to $\overline{PS^2}$, or as $3\overline{PC^2}$ is to $\overline{PQ^2}$, that is, as $3aa$ is to xx ; wherefore $\frac{1}{2}\dot{u} - \dot{u} = \frac{3aa \times \dot{u}}{xx}$ = (substituting instead of $\dot{u}$ its Value $-\frac{2bx}{3}$) $-\frac{2aabbx}{xx}$; therefore $\frac{1}{2}s - u = \frac{2a_2b}{x} - \frac{2a^2b}{r} = \frac{2a^2b}{rx} \times r - x$; wherefore $\frac{1}{2}s - u$ is to u as $\frac{2a^2b}{rx} \times r - x$ is to $\frac{2b}{3} \times r - x$, or as $3aa$ is to rx , or as $3\overline{PC^2}$ is to $\overline{PR} \times \overline{PQ}$, or as $3\overline{PC} \times \overline{PD}$ is to $\overline{PR^2}$. Since therefore $\frac{1}{2}s - u$ is to u as $3\overline{PC} \times \overline{PD}$ is to $\overline{PR^2}$,

it follows, by *Composition*, that $\frac{1}{2} s$ or the Semi-sum of the Solids generated by the Areas A E D C, and $a e D C$ is to u , or the Solid generated by the Sector P R S as $3 P C \times P D + \overline{P R}^2$ is to $\overline{P R}^2$.

Again, since the Solids generated by the Areas P S T, E H K D, and $e b K D$ are to one another as $\overline{P S}^3$, $\overline{P E}^3 - \overline{P D}^3$, and $\overline{P D}^3 - \overline{P e}^3$ respectively, we shall have $\dot{a}$, or the Difference of the Solids generated by the two Areas E H K D, and $e b K D$ to $\dot{u}$ as $\overline{P E}^3 + \overline{P e}^3 - 2 \overline{P D}^3$ to $\overline{P S}^3$, or as $6 P D \times \overline{D E}^2$, is to $\overline{P S}^3$, or as $6 P D$ is to $P S$, or as $6 P C$ is to $P Q$, that is, as $6 a$ is to x ; wherefore $\dot{a} = \frac{6 a \dot{u}}{x}$,

and $\frac{1}{2} \dot{a} = \frac{3 a \dot{u}}{x} = -\frac{2 a b \dot{x}}{x}$, and $\frac{1}{2} \dot{d} = -2 a b \left| \frac{x}{r} \right| = +2 a b \left| \frac{r}{x} \right|$; and therefore the Altitude of a Cylinder

equal to the Semi-difference of the two conchoidal Solids, and whose Base is b , viz. a Circle upon the

Diameter A a will be equal to $2 a \left| \frac{r}{x} \right| = 2 P C \left| \frac{P D}{P C} \right|$:

And thus having found both the Semi-sum and Semi-difference, the Solids themselves will easily be found. Q. E. I.

PROBLEM X.

It is required to measure the conchoidal Areas A E D C, and a e D C, jointly and separately.

ANALYSIS.

Suppose all Things as in the last *Problem*, except that now s represents the Area A E e a, or the Sum of the two Areas A E D C and a e D C, that d is their Difference, and u the Area of the Sector P R S; and the Area E H b e or i will be to the Area P S T or $\dot{u}$ as $\overline{P E^2} - \overline{P e^2}$ to $\overline{P S^2}$, or as $4 P S \times D E$ is to $\overline{P S^2}$, or as $4 P D$ is to $P S$, or as $4 P C$ is to $P Q$, that is, as $4 a$ to x ; wherefore

$$\dot{s} = \frac{4a}{x} \times \dot{u} = \frac{4a}{x} \times -\frac{\frac{1}{2} r r \dot{x}}{\sqrt{r r - x x}} = -\frac{2 a r r \dot{x}}{x \sqrt{r r - x x}},$$

a Fluxion of the *fifth Form*, where $\theta = 0$, and whose Fluent, when made possible, gives $s =$

$$2 a r \left| \frac{r + \sqrt{r r - x x}}{x} = A a \times P C \right| \frac{P S + S Q}{P Q} =$$

$$A a \times P C \left| \frac{P D + D C}{P C} = A a \times C M^*. \quad Q. E. I.$$

Again, $\dot{d}$ or E H K D — e b K D is to $\dot{u}$ as $\overline{P E^2} + \overline{P e^2} - 2 \overline{P D^2}$ is to $\overline{P S^2}$, or as $2 \overline{D E^2}$ is to $\overline{P S^2}$, or as 2 to 1; wherefore $\dot{d} = 2 \dot{u}$, and $\frac{1}{2} d$ is equal to the Area P R S; or if through M a Line as G g is imagined to be drawn parallel to A a, and cutting the Perpendiculars A F and a f in G

* By Construction, see *Harm. Mensur.* p. 25.

and

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and g respectively, $\frac{1}{2}d$ or the Sector PRS will be equal to each of the Triangles MGF or Mgf ; for MG is to GF as PC is to CN ; and therefore as CN is the Measure of the Angle CPD *ad Modulum* PC^* , GF will be the Measure of the same Angle *ad Modulum* MG or PR ; wherefore the Area $AFMC$ or $\frac{1}{2}s + \frac{1}{2}d$ will be equal to the Area $AEDC$, and the Area $afMC$, or $\frac{1}{2}s - \frac{1}{2}d$ will be equal to the Area $aeDC$. Q. E. I.

PROBLEM XI. Fig. 34.

To measure the interior hyperbolic Area ADB :
The Center of the Hyperbola being C , and the conjugate Axis PQ .

ANALYSIS.

CASE I.

Let AB be perpendicular to CE , and call CD , t ; and CP , c ; Ce , x ; and EB , y ; and we shall have, from the Nature of the *Hyperbola*,
 $yy = \frac{cc}{tt} \times \overline{xx - tt}$, and $y = \frac{c}{t} \times \sqrt{xx - tt}$, and $2y \dot{x}$,
or the Fluxion of the Area proposed, $= \frac{2c}{t} \times \dot{x} \times \sqrt{xx - tt}$, a Fluxion of the *fourth Form* where $t=0$; the *rational* Part of whose Fluent is $\frac{cx}{t}$
 $\times \sqrt{xx - tt}$, and the *irrational* Part, when the Terms of the Ratio are made possible, and reduced

* By the Construction.

the same Denomination, is $-ct \left| \frac{x + \sqrt{x^2 - t^2}}{t} \right|$,

both which Parts are equal to nothing, when E coincides with D : So that this Fluent is a true Representative of the Area sought ADB. But

the *rational* Part of this Fluent $\frac{cx}{t} \times \sqrt{x^2 - t^2}$, or xy , is equal to $EB \times CE$, and the *Modulus* ct , or $CD \times CP$, or $EB \times \frac{CD \times CP}{EB}$, or $EB \times \frac{CD \times PQ}{AB}$

is equal to $EB \times CS$, and the Ratio $\frac{x + \sqrt{x^2 - t^2}}{t}$,

or $\frac{x + \frac{ty}{c}}{t}$ is equal to $\frac{CE + CR^*}{CD} = \frac{CD}{ER}$: For

since $\overline{CR}^2 = \frac{t^2 y^2}{c^2} = x^2 - t^2 = \overline{CE}^2 - \overline{CD}^2$, we shall have $\overline{CD}^2 = \overline{CE}^2 - \overline{CR}^2 = \overline{CE + CR} \times \overline{CE - CR}$, whence $CE + CR$ will be to CD as CD is to $CE - CR$ or ER ; whence upon the whole Matter it follows, that the Area ADB is equal to

$EB \times CE - EB \times CS \left| \frac{CD}{ER} = EB \times \overline{CE - CN} = EB \times EN \right.$ equal to the Area of the Triangle ANB. Q. E. I.

* It being $CR : CD :: CD : CS :: AB : PQ$, and $CN = CS$

$\left| \frac{CD}{ER} \right.$ by *Construction*.

CASE II.

Let now the Ordinate AB be supposed to turn upon the Point E , so as to become oblique to the Line CE , and let all the rest of the Ordinates parallel to it be supposed to turn with a like angular Motion upon the several Points of the Axe; all other Things remaining the same as before; and the Figure will still be an *Hyperbola*, because the Expression of the Relation between the several Ordinates, and their respective Abscissae will be the same as before; but the Elements of the Area ADB , or the Intervals of these Ordinates, will now be lessened in the Proportion of the Radius to the Sine of Inclination: But as the Elements of the Triangle ANB will also be lessened in the same Proportion, the Area ADB will still be equal to the Triangle ANB ; and therefore is found in all Cases. Q. E. I.

PROBLEM XII.

To find the Center of Gravity of the interior hyperbolic Area ADB .

ANALYSIS.

CASE I.

Let AB be perpendicular to CE , and let CE be considered as a Lever moveable about the Axis PQ ; and using the Notation of the last Problem, the Fluxion of the *Momentum* of the Area ADB

to turn the Lever will be $\frac{2c}{t} \times \dot{x} \times x \times \sqrt{x x - t t}$, as is evident from *Statics*. But this is a Fluxion of the *third Form*, where $\theta = 1$; and whose Fluent $\frac{2c}{t} \times \frac{\sqrt{x x - t t}}{3} \times x x - t t$ is equal to $\frac{2\overline{CR}^2}{3} \times E B$; and as this Fluent at the Point D is equal to nothing, it will be a just Representative of the *Momentum* of the Area ADB: But as this Area is equal to $E N \times E B$, the Quantity $C Z \times E N \times E B$ will also represent the *Momentum* aforesaid, as is plain from the Nature of the Center of Gravity; wherefore $\frac{2\overline{CR}^2}{3} \times E B = C Z \times E N \times E B$, and $2\overline{CR}^2 = C Z \times 3 E N$, and consequently $C Z$ is to $C R$ as $2 C R$ is to $3 E N$ *. Q. E. I.

CASE II.

Let now the Ordinate A B, and all those that are parallel to it become inclined to the Line CE, as in the second Case of the last Problem; and the several Elements of the Area ADB will now be diminished in their Gravity as well as in their Quantity; but as the Loss will be every where proportionable to the Whole, the Center of Gravity cannot be affected by it, and therefore must be found by the same Methods as before. Q. E. I.

* As in the *Construction*.

PROBLEM XIII. Fig. 35.

To measure the exterior hyperbolic Area APQB.

ANALYSIS.

CASE I.

Let AB be perpendicular to CE, and call CD, t ; CP, c ; CE, x ; and EB, y ; and we shall have $yy = \frac{cc}{tt} \times x \overline{x - t}$; whence proceeding as in the eleventh Problem, we have the Area APQB equal to the Triangle ANB; CN being taken contrary to CE, because now the *irrational* Part of the Fluent is affirmative. Q. E. I.

CASE II.

See Case II. to Problem. Q. E. I.

PROBLEM XIV.

To find the Center of Gravity of the same hyperbolic Area APQB.

ANALYSIS.

Here when CE=0, AB will be equal to PQ, and CR to CD, and the Fluent $\frac{2\overline{CK}^2}{3} \times EB$

(See Problem XII.) to $\frac{2\overline{CD}^2}{3} \times CP = \frac{2CD \times CS}{3} \times EB = \frac{2CT \times CR}{3} \times EB$; and therefore the Mo-

N 4

mentum

sum of the Area APQB will be $\frac{2\overline{CR}^2 - 2CT \times CR}{3}$
 $\times EB = \frac{2TR \times CR}{3} \times EB = CZ \times EN \times EB$:
 Wherefore CZ is to CR as 2TR to 3EN.
 Q. E. I.

PROBLEM XV. Fig. 36.

To find the Surface generated by the Revolution of the Hyperbola AN about its primary Axe AC. Its Vertex being A; its Center C; the Asymptote CB; the Focus F; the Semi-axes AC, AB; XN an Ordinate.

ANALYSIS.

Since CF is to CA as CA to CE (by Construction) or as CB to CG, and since CB=CF, CG will be equal to CA. Call then CA or CG, t ; CF, or CB, m ; AB, c ; CX, x ; and XN, y ;

and we shall have $yy = \frac{cc}{tt} \times x^2 - t^2$, and $y\dot{y} = \frac{c\dot{c}\dot{x}}{tt} \times x$,

and $y^2\dot{y}^2 = \frac{c^2\dot{x}^2}{t^4} \times c^2x^2$. Again, $yy\dot{x}^2 = \frac{c^2\dot{x}^2}{t^2} \times x^2 - t^2$

$= \frac{c^2\dot{x}^2}{t^4} \times t^2x^2 - t^4$; wherefore $yy\dot{x}^2 + y^2\dot{y}^2$, or

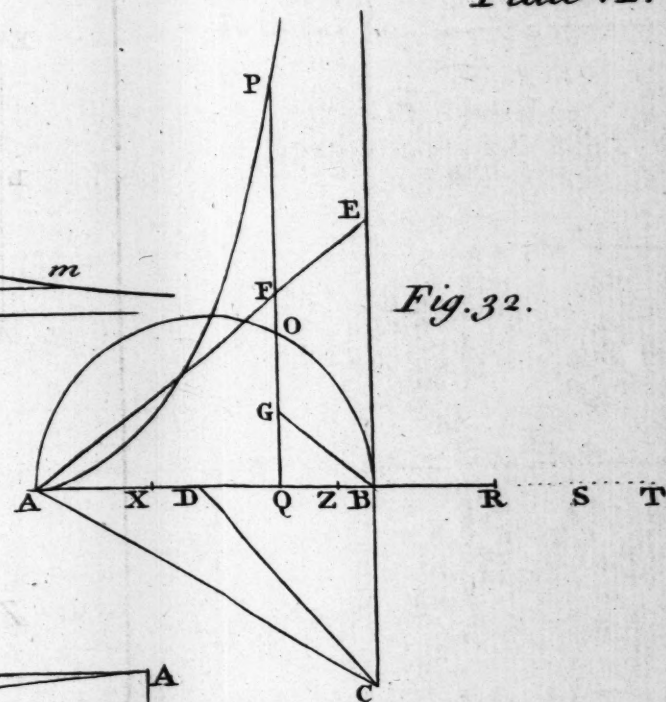
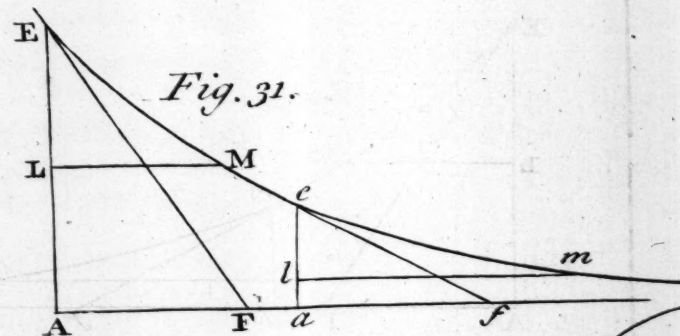
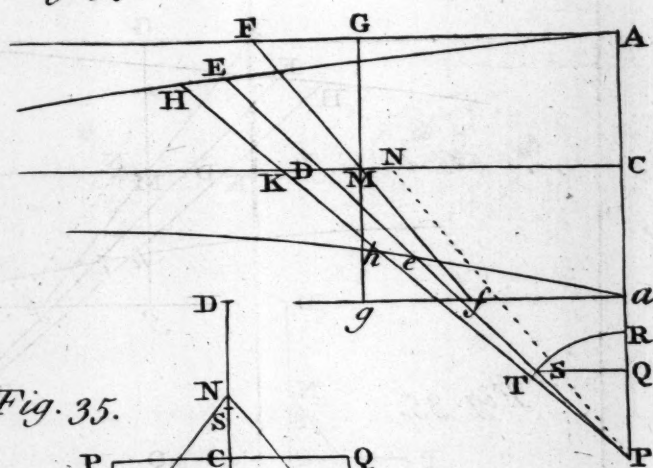
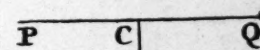
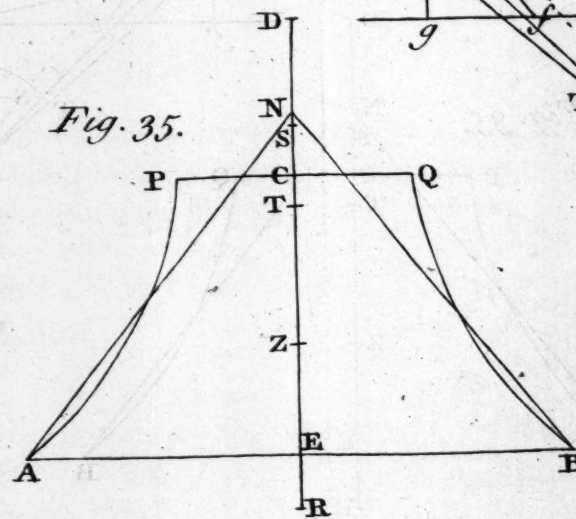
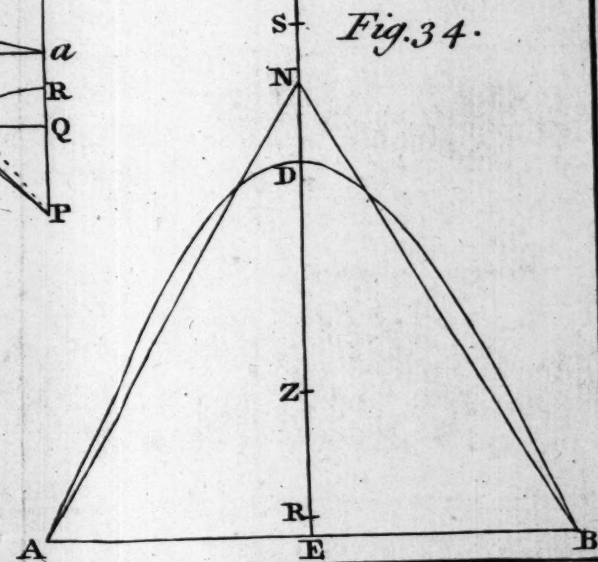
$y^2\dot{x}^2 + \dot{y}^2 = \frac{c^2\dot{x}^2}{t^4} \times c^2x^2 + t^2x^2 - t^4 = \frac{c^2\dot{x}^2}{t^4} \times m^2x^2 - t^4$

$= \frac{c^2m^2\dot{x}^2}{t^4} \times x^2 - \frac{t^4}{m^2} = \frac{c^2\dot{x}^2}{r^2} \times x^2 - r^2$, putting r for

$\frac{tt}{m}$ or CE: Therefore $y\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{c\dot{x}}{r} \times \sqrt{x^2 - r^2}$;

and therefore if we put d for the Semi-circumference

Plate VI.

*Fig. 33.**Fig. 35.**Fig. 34.*

rence of a Circle whose Semidiameter is c or AB ,
 we shall have $\frac{2 dy}{c} \times \sqrt{x^2 + y^2}$, or the Fluxion of
 the Surface sought, $= \frac{2 d\dot{x}}{r} \times \sqrt{x^2 - r^2}$, a Fluxion
 of the *fourth Form*; where $t=0$; the *rational* Part
 of whose Fluent is $\frac{d x}{r} \times \sqrt{x^2 - r^2} = d \times \frac{CX}{CE} \times EZ$
 $= d \times XO$; and the *irrational* Part, when the
 Terms of the Ratio are made possible, and reduced
 to the same Denomination, is $-dr \left| \frac{x + \sqrt{xx - rr}}{r} \right|$
 $= -d \times CE \left| \frac{CZ + ZE}{CE} \right|$. But when X coincides
 with A , CZ (or CX^*) will become CG or CA , and
 XO will become AB , and $\frac{CZ + ZE}{CE}$ will become
 $\frac{CG + GE}{CE}$. Subtract then AB from XO , and
 there will remain KL^* , and subtract $CE \left| \frac{CG + GE}{CE} \right|$
 from $CE \left| \frac{CZ + ZE}{CE} \right|$, and there will remain
 $CE \left| \frac{CZ + ZE}{CG + GE} \right|$ or LM^* , and the Surface sought
 will be $d \times KL - d \times LM$, or $d \times KM$, which
 Surface therefore will be to a Circle whose Semi-
 diameter is AB or c as $d \times KM$ is to $d \times c$, that is,
 as KM to AB . Q. E. I.

* By Construction.

PROBLEM XVI. Fig. 37.

It is required to find the Surface generated by the Revolution of the Hyperbola BN about its secondary Axe AC.

ANALYSIS.

Call CA, t ; CB, c ; CF, m ; CE, r ; CX, x ; XN, y ; and the Semi-circumference of a Circle whose Semidiameter is CB, d ; and then proceeding upon the Equation $yy = \frac{cc}{tt} \times \overline{x^2 + t^2}$, as in the last Problem, we shall have the *rational* Part of the Fluent for determining the Surface equal to $\frac{dx}{r} \times \sqrt{x^2 + r^2} = d \times \frac{CX \times EX}{CE} = d \times KL$, and the *irrational* Part will be $d \times LM$; and as both these Parts, when C and X are coincident, are equal to nothing, the Surface sought will be $d \times \overline{KL + LM} = d \times KM^*$. Q. E. I.

PROBLEM XVII. Fig. 38.

To find the Surface generated by the Revolution of the elliptical Arc BN about its secondary Axe AC.

ANALYSIS.

Apply the Notation of the last Problem to this Scheme, and the Equation $yy = \frac{cc}{tt} \times \overline{tt - xx}$ pursued as in the fifteenth Problem, will give the Surface sought equal to $d \times KM$ as before. Q. E. I.

* By *Constr.* CE:XE::XC:KL and LM=CE $\left| \frac{EX+XC}{CE} \right.$
PRO-

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PROBLEM XVIII. Fig. 39.

To find the Surface generated by the Revolution of the elliptical Arc BN, about its primary Axe AC.

ANALYSIS.

Apply the Notation of the sixteenth Problem to this Scheme, and then proceeding as in the fifteenth Problem, we shall have the *rational* Part of the Fluent for determining the Surface equal to $d \times KL$ as before, which begins to flow with the Surface sought. But the *irrational* Part will be the Measure of an Angle *ad Modulum* $-dr$, whose Secant is to the Radius as r to x , which is the Case of the Angle ECX , and which I thus express $-dr | ECX$, or thus $+dr | -ECX$. But when X coincides with C , and consequently CE with CB , the Angle $-ECX$ becomes $-BCX$; which therefore being subtracted from $-ECX$ leaves $BCX - ECX = BCE = CEX$; and the *irrational* Part, at least so much of it as serves our Purpose will be $dr | CEX = d \times LM$. Q. E. I.

LEMMA Fig. 40.

If two Angles be added together, whereof the *Radius*, *Tangent*, and *Secant* of one are as R , T , and S ; and those of the other as r , t , and s ; those of the Sum will be as $Rr - Tt$, and $Rt + rT$, and Ss , respectively.

DEMON-

DEMONSTRATION.

Draw any Line as AB to represent R the greater of the two *Radii*, and out of it take Ab to represent r the less; and perpendicular to AB , but in contrary Directions draw $BC = T$, and $bc = t$, and joining AC and Ac , let Ac and CB when produced meet in D , and draw CE perpendicular to AD in E : Then it is plain that the Angles BAC and BAD will be equal to the Angles first proposed; that the Sum will be the Angle CAD ; and that the *Radius*, *Tangent*, and *Secant* of this Sum will be as AE , EC , and AC ; or as $AE \times AD$, $EC \times AD$, and $AC \times AD$ respectively. But $EC \times AD$ is equal to $AB \times CD$, since each is equal to twice the Area of the Triangle ACD : And moreover, since by *Prop. XIII. El. II.* $\overline{CD}^2 + 2AE \times AD = \overline{AC}^2 + \overline{AD}^2 = 2\overline{AB}^2 + \overline{BC}^2 + \overline{BD}^2$ if from one Side we take away $\overline{CD}^2$, and from the other $\overline{BC}^2 + 2BC \times BD + \overline{BD}^2$, which by *Prop. IV. El. II.* is equal to it, and lastly, if we take half the rest, we shall have $AE \times AD = \overline{AB}^2 - BC \times BD$. Substitute therefore the latter instead of the former in the Proportion foregoing, and also $AB \times CD$, or $AB \times BC + AB \times BD$ instead of $EC \times AD$, and we shall have the *Radius*, *Tangent*, and *Secant* of the Angle CAD as $\overline{AB}^2 - BC \times BD$, and $AB \times BC + AB \times BD$, and $AC \times AD$, respectively. Put now for $\overline{AB}^2$, $AB \times Ab$,

$AB \times Ab$, or Rr ; for $BC \times BD$, $BC \times bc$, or Tt ; for $AB \times BD$, $AB \times bc$, or Rt ; for $AB \times BC$, $Ab \times BC$, or rT ; and lastly for $AC \times AD$, $AC \times Ac$, or Ss ; and since the Terms are all lessened in the same Proportion, they will still continue in the same Proportion to one another; and the *Radius*, *Tangent*, and *Secant* of the Angle CAD will be as $Rr - Tt$, and $Rt + rT$, and Ss , respectively. Q. E. D.

SCHOLIUM.

For the better Application of the foregoing *Theorem* it will be necessary to observe, *first*, That if the *Radius*, *Tangent*, and *Secant* of any Angle have all the same Sign before them, the Angle will be *acute*. That if the Sign of the *Radius* be different from those of the *Tangent* and *Secant*, it will be *obtuse*. That if the Sign of the *Tangent* be different from those of the *Radius* and *Secant*, the Angle will be *acute* and *negative*. That if the Sign of the *Secant* be different from those of the *Radius* and *Tangent*, it will be *negative* and *obtuse*: And lastly, That if the *Radius* be *nothing*, or the *Tangent* and *Secant*, *infinite*, the Angle will be a right one. These are the Marks whereby Angles are to be distinguished one from another, and which must be duly observed, not only before Addition, in order to put proper Signs before the *Radius's*, *Tangents*, and *Secants* of the Angles to be added, but also afterwards, in order to judge aright of the Aggregate.

Secondly, It must be observed, that if the *Radius*, *Tangent*, and *Secant* of any Angle be as R , T , and S , those of its Complement to a right one will be as T , R , and S ; and those of its Complement to two right ones as R , $-T$, and $-S$ respectively. Not that I would here be understood as if the *Tangent* and *Secant* of this Complement were always to be *negative*, but that they must always be contrary to those of the Angle first proposed.

Many other Observations might be drawn from the foregoing Principle; as that we are furnished with a Method for subtracting one Angle from another*; for investigating a general Theorem whereby any Number of Angles may be added together; for multiplying, or dividing an Angle by any given Number whatsoever, &c. But these, however useful they may be in themselves, being not so much for our present Purpose, we shall not insist upon them, but proceed to apply what has already been taken Notice of.

* By changing the Sign of t in the *Lemma*.

PROBLEM XIX. Fig. 41.

It is required to determine the Law of Attraction wherewith a Spheroid attracts any Particle of Matter placed in the Direction of its Axis produced; supposing the whole Spheroid to be of the same Density, and to consist of Particles which considered by themselves attract at all Distances with Forces reciprocally proportionable to the Squares of their Distances.

CONSTRUCTION.

Let ADBE be the generating *Ellipse*, AB the greater Axe, DE the lesser, C the Center, and F the Focus; and if AB be the Axis of the *oblong* Spheroid in the Direction whereof the attracted Particle Q is supposed to be placed; then in the Angle FCD inscribe a Line as FP equal in Length to the Distance of the Particle from the Center of the Spheroid: But if the Spheroid is a *prolate* one whose Axis is DE, and the Particle be supposed to be placed in the Direction of that Axis; then let FP be a Line drawn from the Focus to the Center of the Particle itself; and in the Line CF produced, in the first Case, take CG equal to the Measure of the Ratio between $PF + FC$ and PC: But in the latter Case, set off upon the Line CF the Part CG equal to the Measure of the Angle CPF; and let the Distance of the attracted Particle from the Center C be the

Modulus

Modulus in both Cases, namely, PF in the former Case, and PC in the latter; and the Attraction of the Spheroid will be as the Quantity of Matter in the Spheroid and the Line FG directly, and the Cube of the Line CF inversely.

ANALYSIS.

CASE I.

Let Q be the attracted Particle in the Direction of the Axis BA produced, and assuming any where in that Axis AB, a Point as X, draw the Ordinate XN, and join NQ. Call CA, t ; CD, c ; CF, m ; CQ, a ; XQ, x ; and NQ, z ;

and we shall have $\overline{XN}^2 = \frac{\overline{CD}^2}{\overline{CA}^2} \times \overline{CA}^2 - \overline{CX}^2 =$

$$\frac{c^2 t t - c^2 a^2 + 2 c^2 a x - c^2 x^2}{t t}, \text{ and consequently } z^2 =$$

$$\frac{c^2 t^2 - c^2 a^2 + 2 c^2 a x - c^2 x^2}{t t} + x x = \frac{c^2 t^2 - c^2 a^2 + 2 c^2 a x + m^2 x^2}{t t},$$

$$\text{and } z = \frac{\sqrt{c^2 t^2 - c^2 a^2 + 2 c^2 a x + m^2 x^2}}{t}. \text{ But by the}$$

ninetieth and ninety first *Prop.* of the first Book of the *Principia*, the Force wherewith any circular elementary *Lamina* whose Semidiameter is XN attracts the Particle Q will be as

$$\dot{x} - \frac{x \dot{x}}{z}; \text{ and therefore the Fluent of the Fluxion}$$

$$\dot{x} - \frac{t x \dot{x}}{\sqrt{c^2 t^2 - c^2 a^2 + 2 c^2 a x + m^2 x^2}} \text{ (I mean so much}$$

of

of it as lies between A and B) will represent the whole Force of Attraction required. But $\dot{x}$ is a simple Fluxion whose Fluent is x , and

$\frac{-t x \dot{x}}{\sqrt{c^2 t^2 - c^2 a^2 + 2 c^2 a x - m^2 x^2}}$ is a Fluxion of the sixteenth Form, where $\theta=2$; the rational Part of whose Fluent is $\frac{-t^2 z}{m^2}$, and the irrational Part the

Measure of a Ratio *ad Modulum* $\frac{c_2 t a}{m^3}$, whose Antecedent is $m + \frac{c^2 a + m^2 x}{t z} = \frac{t m z + c^2 a + m^2 x}{t z}$, and

whose Consequent is $\frac{\sqrt{c^4 a^2 + c^2 m^2 a^2 - c^2 t^2 m^2}}{t z} =$

$\frac{\sqrt{c^2 t^2 a^2 - c^2 t^2 m^2}}{t z} = \frac{c t \times \sqrt{a^2 - m^2}}{t z}$; and therefore the irrational Part of this Fluent will be $\frac{c_2 t a}{m^3} \left| \frac{t m z + c c a + m^2 x}{c t \sqrt{a^2 - m^2}} \right|$

Now the Quantity $x - \frac{t^2 z}{m^2}$, when X coincides with A will be equal to $1 - \frac{t^2}{m^2} \times a - t = -\frac{c_2}{m^2} \times a - t$, and when X coincides with B will be equal to $-\frac{c c}{m^2} \times a + t$; and the former subtracted from the latter leaves $-\frac{2 c^2 t}{m^2}$, or $-\frac{2 c^2 t m}{m^3}$ for our Purpose.

Again, the Ratio $\frac{t m z + c^2 a + m^2 x}{c t \sqrt{a^2 - m^2}}$ at the Point
O A be-

A becomes $\frac{t m a - t^2 m + c^2 a + m^2 a - m^2 t}{c t \sqrt{a^2 - m^2}} =$
 $\frac{t m a - t^2 m + t^2 a - m^2 t}{c t \sqrt{a^2 - m^2}} = \frac{a - m \times t + m}{c \sqrt{a^2 - m^2}}$; and at the

Point B the aforefaid Ratio becomes $\frac{a + m \times t + m}{c \sqrt{a^2 - m^2}}$;

and the former subtracted from the latter leaves $\frac{a + m}{a - m}$; and therefore so much of the *irrational*

Part as lies between A and B will be $\frac{c^2 t a}{m^3} \left| \frac{a + m}{a - m} \right|$

$$= \frac{c c t}{m^3} \text{PF} \left| \frac{\text{PF} + \text{FC}}{\text{PF} - \text{FC}} \right| = \frac{2 c^2 t}{m^3} \text{PF} \left| \frac{\text{PF} + \text{FC}}{\text{PC}} \right| = \frac{2 c^2 t}{m^3} \times \text{CG} :$$

But the other Part of the Fluent was found to be

$$- \frac{2 c^2 t m}{m^3} \text{ or } - \frac{2 c^2 t}{m^3} \times \text{CF} ; \text{ and therefore the}$$

Attraction of the Spheroid will be $\frac{2 c^2 t}{m^3} \times \text{FG}$, or

as its Magnitude and the Line FG directly, and the Cube of the Line CF inversely.

Hitherto we have supposed the Density of the Spheroid to be given; but now if we suppose that to vary, and all other Things to continue as before, the Attraction will be as the Density: And therefore universally it will be as the Density and Magnitude of the Spheroid, and the Line FG directly, and the Cube of the Line CF inversely. But the Quantity of Matter contained in any Body is as its Magnitude and Density together: And therefore now the Attraction of the Spheroid will

be as its Quantity of Matter and the Line FG directly, and the Cube of the Line CF inversely:
Q. E. I.

CASE II.

Let now P be the attracted Particle in the Axis ED produced, and from any Point as x in that Axis draw the Ordinate xn , and join nP . Call CD, t ; CA, c ; CF, m ; CP, a ; xP , x ; and nP , z ; and because now $m^2 = c^2 - t^2$, and consequently $z^2 = \frac{c^2 t^2 - c^2 a^2 + 2 c^2 a x + m^2 x^2}{t^2}$, the rational Part of the Fluent for determining the Attraction of the Spheroid will now be $x + \frac{t^2 z}{m^2}$, whereof $\frac{2 c^2 t m}{m^3}$ is for our Purpose. And moreover the Measure of an Angle *ad Modulum* $\frac{c^2 t a}{m^3}$, whose Radius, Tangent, and Secant will be to one another as the Quantities $t m z$, $c^2 a - m^2 x$, and $c t \sqrt{a^2 + m^2}$; which at the Point D will be as $t m a - t^2 m$, $c^2 a - m^2 a + m^2 t$, and $c t \sqrt{a^2 + m^2}$; or as $t m a - t^2 m$, $t^2 a + m^2 t$, and $c t \sqrt{a^2 + m^2}$; or as $a m - m t$, $t a + m^2$, and $\sqrt{t^2 + m^2} \times \sqrt{a^2 + m^2}$. Let there be now two Angles, whereof the Radius, Tangent, and Secant of one, which we shall call p , are as t , m , and $\sqrt{t^2 + m^2}$, and those of the other, which we shall call q , as a , m , and $\sqrt{a^2 + m^2}$; and (by the last Lemma) those of the Angle $p - q$ will be as $t a + m^2$,
O 2 and

and $am - mt$, and $\sqrt{t^2 + m^2} \times \sqrt{a^2 + m^2}$; and those of its Complement $1 - p + q$, putting 1 for a right Angle, will be as $am - mt$, and $ta + m^2$, and $\sqrt{t^2 + m^2} \times \sqrt{a^2 + m^2}$ respectively: So that the *irrational* Part of the Fluent at the Point D is $\frac{c^2 t a}{m^3} \Big| 1 - p + q$; and by a like Computation, at the Point E it will be found to be $\frac{c^2 t a}{m^3} \Big| 1 - p - q$; and the former subtracted from the latter leaves $\frac{c^2 t a}{m^3} \Big| -2q$, or $-\frac{2 c^2 t a}{m^3} \Big| q$, or $-\frac{2 c^2 t a}{m^3} \Big| \text{C P F}$, or $-\frac{2 c^2 t}{m^3} \times \text{C G}$; which joined with the Part above found $\frac{2 c^2 t}{m^3} \times \text{C F}$ will make $\frac{2 c^2 t}{m^3} \times \text{F G}$, for the Attraction required: Whence it follows, that in this Case also the Attraction of the Spheroid will be as the Quantity of Matter, and the Line F G directly, and the Cube of the Line C F inverfely. Q. E. I.

PROBLEM XX.

Supposing all Things as in the last Problem; let it be required to compare the Attraction of the Spheroid with that of a Sphere of the same Density, and upon the same Axis.

In the former Case of the last Problem, the Line C G was taken equal to $a \Big| \frac{a+m}{\sqrt{a^2 - m^2}} = \frac{1}{2} a \Big| \frac{a+m}{a-m}$ = (by the fifth Coroll. of the first Prop. of the

Logo-

Logometria) $m + \frac{m^3}{3a^2} \mathcal{E}c$. And in the second *Case* C G will be equal to $m - \frac{m^3}{3a^2} \mathcal{E}c$. (See the Series for finding an Arc from the Tangent given.) Wherefore in both Cases F G will be equal to $\frac{m^3}{3a^2} \mathcal{E}c$. and $\frac{2cct}{m^3} \times F G = \frac{2cct}{3a^2} \mathcal{E}c$. Supposing now the Quantities a and t to continue, let the Quantity m , and all its Adherents signified by the Note $\mathcal{E}c$. vanish; and the Spheroid will, by this Means, be changed into a Sphere upon the same Axis, whose Attraction will be $\frac{2t^3}{3a^2}$, whence it follows,

First, That the Attraction of a Sphere at all Distances will be as the Quantity of Matter therein contained directly, and as the Square of those Distances reciprocally: And

Secondly, That the Attraction of the Spheroid is to that of a Sphere of the same Density and upon the same Axe as $\frac{2cct}{m^3} \times F G$ to $\frac{2t^3}{3a^2}$, or as $3c^2a^2 \times F G$ to $t^2 \times m^3$; and consequently at either Extremity of the Axis, the Attraction of the Spheroid will be to that of the Sphere as $3c^2 \times F G$ to m^3 . Q. E. I.

PROBLEM XXI*. Plate VII.

To find all the Variety of Trajectories, which a Body projected from a given Place with a given Velocity, and in a given Direction may describe when acted upon by a centripetal Force reciprocally proportionable to the Cube of its Distance from the Center of Attraction.

SOLUTION.

CASE I. Fig. 42.

Let SP, SQ, PQ, &c. be considered as indeterminate, and draw the Line Spr infinitely near the Line $S_{\pi}PR$, cutting the Line PpQ_2 and the Arc ARR , in p and r respectively; and taking $S_{\pi}=Sp$, join $p\pi$. Call SC , a ; SP , x ; and if the centripetal Force at any given Distance d be called $2d$, its Force at any other Altitude x will be $\frac{2d^4}{x^3}$, and (by the thirty ninth and fortieth

Prop. of the first Book of the *Principia*) the Velocity of the Body at the same Altitude x will be as

$\frac{dd}{ax} \sqrt{aa - xx}$, or as $\frac{\sqrt{a^2 - x^2}}{x}$, and the Perpendicular

SQ_2 which is always reciprocally as this Velocity, will be as $\frac{x}{\sqrt{a^2 - x^2}}$. Let us now assume

* See Mr. Cotes's Constructions in *Harm. Mensurarum*, p. 31. et seqq.

a given

Fig. 36.

Fig. 37.

Fig 38

Fig. 39.

Fig. 40.

Fig. 41.

a given Quantity as b of such a Magnitude that at the first setting out the Perpendicular SQ may

be equal to $\frac{bx}{\sqrt{a^2-x^2}}$, and it will be so ever after-

wards, and we shall have $PQ = x \times \sqrt{\frac{a^2-b^2-x^2}{a^2-x^2}}$,

and $QT = a \times \sqrt{\frac{a^2-b^2-x^2}{a^2-x^2}}$, and $QT^2 =$

$\frac{a^4-a^2b^2-a^2x^2}{a^2-x^2}$, and $SQ^2+QT^2=a^2-b^2$, and

and $\sqrt{SQ^2+QT^2}$ or $SR = \sqrt{aa-bb}$ a given Quantity, which for the future we shall call n .

Now since $P\pi$ is to πp as PQ is to QS , that is, as $\sqrt{a^2-b^2-x^2}$ to b , or as $\sqrt{nn-xx}$ to b ; and

since πp is to Rr as SP is to SR , or x to n , it follows, joining the Ratio's together, that $P\pi$ or

$-\dot{x}$ is to Rr as $x \times \sqrt{n^2-x^2}$ is to bn ; and therefore Rr , or the Fluxion of the Arc AR will be

equal to $\frac{-bn\dot{x}}{x\sqrt{n^2-x^2}}$ a Fluxion of the fifth Form,

where $\theta=0$, and whose Fluent is $b \left| \frac{n+\sqrt{n^2-x^2}}{x} \right|$

$=SM \left| \frac{SE+ED}{SD} \right|$; making $SM=b=\sqrt{SC^2-SR^2}$,

and taking SD always equal to the Altitude SP .

From which Construction it follows,

First, That the Trajectory AP will touch the Circle AR in the Point A .

Secondly, That the Altitude SA or n is the highest to which the Body can ascend; for should it ascend higher, the Quantities QT , &c. wherein $\sqrt{n^2-x^2}$ is concerned, would become impossible.

Thirdly, That therefore the Point A is the *Apsis summa* of this Trajectory, and that it has no other Apsis but the Point A.

Fourthly, That the Ratio $\frac{SE + ED}{SD}$, and its Measure AR, and the Angle ASP may be of any Magnitude from Nothing to Infinity, and consequently, that the Body P in its oblique Descent must make an infinite Number of Revolutions before it can come into the Center S.

Fifthly, That therefore the Spiral AP must necessarily cut the Line SA in an infinite Number of Points, which may also be said of the Spiral on the other Side of the Line SA; and as these two Curves are similar, they must necessarily cut the Line SA in the same Points, which therefore must be the Points where they cut one another: So that the *Nodes* must all be in the Line SA; and thus much for the Form of the Trajectory.

Now as for the Area SAP, since $Rr = \frac{-bn\dot{x}}{x\sqrt{n^2-x^2}}$, and consequently the Area $SRr = \frac{-\frac{1}{2}bn^2\dot{x}}{x\sqrt{n^2-x^2}}$, the Area SPp, or the Fluxion of the Area SAP will be equal to $\frac{-\frac{1}{2}bx\dot{x}}{\sqrt{nn-xx}}$, a Fluxion of the same fifth Form, where $t=1$, whose Fluent $\frac{1}{2}b \times \sqrt{n^2-x^2}$, or $\frac{1}{2}SM \times ED$ will be equal to the Area SAP. Q. E. I.

CASE

CASE II. Fig. 43.

Draw SQ perpendicular to the Tangent PM , and the Velocity in this Case will be reciprocally, and consequently the Perpendicular SQ directly as SP ; wherefore the Triangle SPQ will always be similar to itself, and the Angle SPM always the same; therefore the Curve in this Case will be the equi-angular Spiral, whose peculiar Property it is, that the intercepted Arc PX shall be the Measure of the Ratio between SZ and SP , or SD and SP , *ad Modulum* SM ; whence it follows also, calling SZ , x , that the Fluxion of the Arc

PX is equal to $\frac{SM \dot{x}}{x}$; and the Fluxion of the Sector SPX equal to $\frac{1}{2} SM \times \frac{SX \dot{x}}{x}$, and the

Fluxion of the Area SPZ equal to $\frac{\frac{1}{2} SM}{SX} \times x \dot{x}$, whose Fluent $\frac{\frac{1}{2} SM}{SX} \times \frac{1}{2} x x$ gives the Area SPZ

equal to $\frac{\frac{1}{2} SM}{SX} \times \frac{\overline{SZ^2 - SP^2}}{2}$; or which is all one, the Area SPZ will be to $\frac{\overline{SZ^2 - SP^2}}{2}$ as $\frac{1}{2} SM$ is to SX or SP .

CASE III. Fig. 44.

Here applying the Notation of the first Case to this Scheme, the Velocity of the Body at the

Height x will be as $\frac{\sqrt{a^2 + x^2}}{x}$, and $\sqrt{a^2 - b^2}$ or n

will

will be equal to $\sqrt{QT^2 - SQ^2}$: Whence proceeding as in the first Case, we shall have the Arc $BR = SM \left| \frac{SE + ED}{SD} \right|$; and the Area described by the Line SP in any given Time during the oblique Ascent of the Body P will be equal to a Rectangle under $\frac{1}{2} SM$, and the Increment of the Line DE generated in the same time. Q. E. I.

CASE IV. Fig. 47.

Draw Sp infinitely near SP, cutting the Curve PpZ in p, and the Arc PB in π , and upon the Center S, and with the Radius SC sweep an Arc of a Circle cutting the Lines SP, Sp, and SB, produced if need be in K, L, and M respectively; and since in this Case $\sqrt{QT^2 - SQ^2}$, or $\sqrt{a^2 - b^2} = 0$, and consequently $a = b$, we shall have πp to πP (that is PQ to QS) as x to a : But πP is to KL also as x to a ; wherefore putting both Ratio's together πp or $-\dot{x}$ will be to KL as xx is to aa ; wherefore KL, or the Fluxion of the Arc KLM will be $-\frac{a^2 \dot{x}}{xx}$; and therefore if we consider the Line SB as a *Radius primus* to which the Body P will not come but after an infinite Ascent, the Arc KLM will be equal to $\frac{aa}{x}$, and consequently the Arc PB will be equal to a or SC; and the same may be said of any other Arc DZ; and therefore the

the Nature of the Curve P Z is sufficiently determined.

Again, the Fluxion of the Area SKM is $-\frac{\frac{1}{2}a^3\dot{x}}{xx}$; and therefore the Fluxion of the Area SPZ will be $\frac{1}{2}a\dot{x}$, whose Fluent $\frac{1}{2}ax$ gives the Area $SPZ = \frac{1}{2}SC \times \overline{SZ} - SP$. Q. E. I.

CASE V. Fig. 46.

Call $\sqrt{SC^2 - QT^2}$, or $\sqrt{bb - aa}$, n ; and we shall have now the Fluxion of the Arc AR = $\frac{bn\dot{x}}{x\sqrt{x^2 - n^2}}$, and the Arc itself equal to the Measure of an Angle *ad Modulum* b , whose Radius, Tangent, and Secant are to one another as n , $\sqrt{x^2 - n^2}$, and x : = SM | D S E, taking SM or b equal to $\sqrt{a^2 + n^2} = \sqrt{SC^2 + SR^2}$, and SD = SP; whence it follows,

First, That the Point A will be the only Apfis of the Trajectory A P, and that the Body P cannot possibly descend lower than the Altitude SA or n , because if it should, the Quantity $\sqrt{x^2 - n^2}$ would become impossible.

Secondly, That the Angle A S P will always be to the Angle A S E as S M to S A.

Thirdly, And since the Angle A S E can never exceed a right one, that the greatest Angle A S P, when the Body P has ascended *ad infinitum* will be to a right one as S M to S A.

As

As for the Area SAP , that will be found equal to $\frac{1}{2} SM \times DE$, as in the first Case. Q. E. I.

If in the *second Case*, the Angle SPM is a right one, the equi-angular Spiral described in that Case will become a Circle, and the Body P will now describe a Circle at the Distance SP with the same Velocity with which it before described the equi-angular Spiral, that is, with the Velocity which it would acquire in falling from an infinite Height to the Altitude SP ; and therefore the Velocity of the Body P in all Cases at the indeterminate Height SP , will be to the Velocity of a Body describing a Circle at the same Distance, and with the same centripetal Force, as it is to the Velocity acquired in falling from an infinite Height to the same Altitude SP , that is, in the first Case as $\frac{dd}{ax} \times \sqrt{a^2 - x^2}$ is to $\frac{dd}{x}$, or as $\sqrt{a^2 - x^2}$, is to a , or as $\sqrt{SC^2 - SP^2}$ is to SC ; and in the third, fourth, and fifth Cases, as $\sqrt{SC^2 + SP^2}$ is to SC .

Two PROBLEMS: Of the OSCILLATIONS of a PENDULUM; and the RECTIFICATION of an ELLIPSE.

LEMMA I.

Let there be two Series's proposed, out of which a third is to be formed, by multiplying the Terms of one into the respective Terms of the other; and let the Terms of the first Series arise from a continual Multiplication of the Quantities $a, b, \&c.$ as also those of the others from a continual Multiplication of the Quantities $c, d, \&c.$ I say then, that the Terms of the third Series will arise from a continual Multiplication of the Quantities $ac, bd, \&c.$ For the two first Terms of the first Series will be a and ab , those of the second c , and cd , and those of the third ac , and $acbd, \&c.$

LEMMA II.

Let b be a constant Quantity, and x and y variable ones; and let $\sqrt{bx - xx} = y$; let $\frac{\dot{x}}{y} = \dot{p}$, $x\dot{p} = \dot{Q}$, $x\dot{Q} = \dot{R}$, $x\dot{R} = \dot{S}$, $x\dot{S} = \dot{T}$, $\&c.$ Then we shall have $\frac{1}{2}b\dot{P} - y = \dot{Q}$, $\frac{3}{4}b\dot{Q} - \frac{1}{2}y\dot{x} = \dot{R}$, $\frac{5}{8}b\dot{R} - \frac{1}{4}y\dot{x}\dot{x} = \dot{S}$, $\frac{7}{8}b\dot{S} - \frac{1}{4}y\dot{x}^3 = \dot{T}$ $\&c.$

DEMON-

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DEMONSTRATION.

If $d \dot{z} z^{\theta-1} \times \overline{e + f z^{\lambda-1}} = \dot{P}$, and $z^{\lambda} \dot{P} = \dot{Q}$, we shall have the following Equation $\theta e P + \overline{\theta + \lambda} . f Q = \frac{d z^{\theta}}{\eta} \times \overline{e + f z^{\lambda}}$. See *Cotes's Logometria*, p. 66. *Theor.* I. But in our Case $\frac{\dot{x}}{\sqrt{b x - x x}}$, or $\dot{x} \times x^{-\frac{1}{2}} \times \overline{b - x}^{-\frac{1}{2}} = \dot{P}$; therefore according to our Notation $z = x$, $d = 1$, $e = b$, $f = -1$, $\eta = 1$, $\theta = \frac{1}{2}$, $\lambda = \frac{1}{2}$, and $x \dot{P} = \dot{Q}$: Therefore the above-quoted Equation in our Language furnishes us with the following ones $\frac{1}{2} b P - Q = y$, $\frac{1}{2} b Q - 2 R = y x$, $\frac{1}{2} b R - 3 S = y x x$, $\frac{1}{2} b S - 4 T = y x^3$, &c. Whence from P may be derived the Quantities Q, R, S, T as in the *Lemma*.

SCHOLIUM.

From the same Principle, if $\frac{\dot{x}}{\sqrt{1 - x x}} = \dot{P}$, $x^2 \dot{P} = \dot{Q}$, $x^2 \dot{Q} = \dot{R}$, $x^2 \dot{R} = \dot{S}$, $x^2 \dot{S} = \dot{T}$, &c. we shall have $\frac{P - x \times \sqrt{1 - x x}}{2} = Q$, $\frac{3Q - x^3 \times \sqrt{1 - x x}}{4} = R$, $\frac{5R - x^5 \times \sqrt{1 - x x}}{6} = S$, $\frac{7S - x^7 \times \sqrt{1 - x x}}{8} = T$, &c. which may serve for the Rectification of the *Ellipse*.

P R O-

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PROBLEM I. Fig. 47.

To find the Time of a single Oscillation of a Pendulum about the Center C made in any given circular Arc HAL, whose lowest Point is A.

Let the horizontal Line HL cut the Line CA in N, and let the Pendulum be supposed ascending in the Arc ADL at the Point D ready to describe the infinitely small Arc Dd, and draw the Line DEB perpendicular to ABC, meeting in E a Semicircle whose Diameter is NA. Let 2CA (or the Diameter of the Circle described by the Pendulum) be equal to d , its Semi-circumference equal to c , $NA = b$, $AB = x$, $BE = y$, and the Arc $AE = v$: Then will the Arc

$$Dd = \frac{\frac{1}{2} d x}{\sqrt{dx - xx}}, \text{ and the Line } NB = b - x \text{ or } \frac{yy}{x},$$

and the Velocity acquired in falling through the Space d will be to the Velocity acquired in falling through the Space NB, or to the Velocity of the

Pendulum in the Point D, as $\sqrt{d}$ is to $\frac{y}{\sqrt{x}}$, or as

$\sqrt{dx}$ to y : Therefore if $\sqrt{dx}$ represents the Velocity acquired in falling through the Space d , y will represent the Velocity of the Pendulum in the Point D. Moreover the Time wherein the Space $2d$ is uniformly described by the Velocity $\sqrt{dx}$ will be to the Time wherein the Space Dd is uni-

formly described by the Velocity y , as $\frac{2d}{\sqrt{dx}}$ is to $\frac{Dd}{y}$, that

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that is, as $\frac{2d}{\sqrt{dx}}$ is to $\frac{\frac{1}{2}d\dot{x}}{y \times \sqrt{dx - xx}}$, or as 1 is to $\frac{\frac{1}{4}\dot{x}}{y \times \sqrt{1 - \frac{x}{d}}}$, or as 1 to $1 - \frac{x}{d} \times \frac{\dot{x}}{y}$. Therefore

if t be the Time of the Descent of a heavy Body through the Space d , and consequently the Time wherein the Space $2d$ will be uniformly described with the Velocity $\sqrt{dx}$; then the Time of the Ascent of the Pendulum through the Arc Dd will be $1 - \frac{x}{d} \times \frac{\dot{x}}{y}$. Let the Quantity $1 - \frac{x}{d} \times \frac{\dot{x}}{y}$

be thrown into an infinite Series, and then we shall have the Time of Ascent from D to d equal

to $\frac{1}{4}t \times \frac{\dot{x}}{y} + \frac{1}{4}t \times \frac{x\dot{x}}{2dy} + \frac{1}{4}t \times \frac{3x^2\dot{x}}{8d^2y} + \frac{1}{4}t \times \frac{5x^3\dot{x}}{16d^3y} + \frac{1}{4}t \times \frac{35x^4\dot{x}}{128d^4y}$, &c. Let $\frac{\dot{x}}{y}$ or $\frac{2v}{b} = P$, $xP = Q$,

$xQ = R$, $xR = S$, $xS = T$, &c. then will the Time of Ascent from A to D be equal to $\frac{1}{4}t \times P +$

$\frac{1}{4}t \times \frac{1}{2d} \times Q + \frac{1}{4}t \times \frac{3}{8dd} \times R + \frac{1}{4}t \times \frac{5}{16d^3} \times S + \frac{1}{4}t \times \frac{35}{128d^4} \times T$, &c. But $\frac{2v}{b} = P$, and by the

foregoing Lemma $\frac{1}{2}bP - y = Q$, $\frac{1}{4}bQ - \frac{1}{2}yx = R$, $\frac{1}{2}bR - \frac{1}{3}yx^2 = S$, $\frac{7}{8}bS - \frac{1}{4}yx^3 = T$, &c. Therefore

at the Point L , where $\frac{2v}{b} = \frac{2c}{d}$, and y vanishes, we have $\frac{2c}{d} = P$, $\frac{1}{2}bP = Q$, $\frac{1}{4}bQ = R$, $\frac{1}{2}bR = S$,

$\frac{7}{8}bS = T$,

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$\frac{1}{4}bS = T$, &c. Therefore in this Case, the Quantities P, Q, R, S, T, &c. arise from a continual Multiplication of these, viz. $\frac{2c}{d}$, $\frac{1}{2}b$, $\frac{3}{4}b$, $\frac{5}{8}b$, $\frac{7}{8}b$, &c. But the Terms of the other Series into which the Quantities P, Q, R, S, T, &c. are to be respectively multiplied, viz. $\frac{1}{4}t$, $\frac{1}{4}t \times \frac{1}{2d}$, $\frac{1}{4}t \times \frac{3}{8dd}$, $\frac{1}{4}t \times \frac{5}{16d^3}$, $\frac{1}{4}t \times \frac{35}{128d^4}$, &c. arise from a continual Multiplication of these, $\frac{1}{4}t$, $\frac{1}{2d}$, $\frac{3}{4d}$, $\frac{5}{6d}$, $\frac{7}{8d}$, &c. If therefore a Series be formed out of both these by a continual Multiplication of $\frac{2c}{d} \times \frac{1}{4}t$, or $\frac{1}{2} \frac{tc}{d}$, $\frac{1}{4} \frac{b}{d}$, $\frac{3}{16} \frac{b}{d}$, $\frac{5}{32} \frac{b}{d}$, $\frac{7}{64} \frac{b}{d}$, &c. the Sum of the Series will exhibit the Time of Ascent from A to L, and if the first Term, instead of $\frac{1}{2} \frac{tc}{d}$ be made equal to $\frac{tc}{d}$, and the rest derived as above, the Series will exhibit the Time of an entire Oscillation through the Arc HAL, and the first Term $\frac{tc}{d}$ will be the Time of the least Oscillation of that Pendulum. Q. E. I.

COROLLARY.

The Time of the Descent of a heavy Body through Half the Length of any Pendulum falling
P from

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from Rest is to the Time of a least Oscillation of that Pendulum as the Diameter of a Circle is to the Circumference.

SCHOLIUM.

If $2v$ be an Arc whose Sagitta is b , and Diameter d , we have from the Doctrine of Fluxions this Equation $\frac{b}{d} = \frac{v^2}{d^2} - \frac{4}{3 \times 4} \frac{v^4}{d^4} + \frac{4 \times 4}{3 \times 4 \times 5 \times 6} \frac{v^6}{d^6} - \frac{4 \times 4 \times 4}{3 \times 4 \times 5 \times 6 \times 7 \times 8} \frac{v^8}{d^8}$, &c. whence the Series in the first Problem, viz. $\frac{1}{4} \frac{b}{d} + \frac{1}{4} \times \frac{9}{16} \frac{b^2}{d^2} + \frac{1}{4} \times \frac{9}{16} \times \frac{25}{36} \frac{b^3}{d^3} + \frac{1}{4} \times \frac{9}{16} \times \frac{25}{36} \times \frac{49}{64} \frac{b^4}{d^4}$, &c. becomes equal to this, viz. $\frac{1}{4} \frac{v^2}{d^2} + \frac{11}{192} \frac{v^4}{d^4} + \frac{173}{11520} \frac{v^6}{d^6} + \frac{22931}{5160960} \frac{v^8}{d^8}$, &c. whence may be deduced the following Theorem.

Make $A = ,4112335$ in decimal Parts of one Second of Time, its Logarithm $-1,6140885$, $B = ,000001794210$, its Logarithm $-6,2538748$, $C = ,000000000008953766$, its Logarithm $-12,9520057$, $D = ,00000000000000005043656$, its Logarithm $-17,7027454$, and let n be the Number of Degrees, and Parts of a Degree contained in any Arc: I say then that the Time of any Number of least Vibrations of any Pendulum will be to the Time of the same Number of Vibrations made by the same Pendulum in an Arc of n Degrees,

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n Degrees, as one Day is to one Day $+ A n^2$
 $+ B n^4 + C n^6 + D n^8$, &c. Seconds. As for Ex-
 ample. Let the Proportion be required in an
 Arc of 120 Degrees. Here $n = 120$, $A n^2 =$
 5921,8 Seconds, $B n^4 = 372,0$ Seconds, $C n^6 =$
 26,7 Seconds, $D n^8 = 2,2$ Seconds. The
 whole Augmentation amounts to 1 Hour, 5921,8
 45 Minutes, 23 Seconds: Therefore if 372,0
 it was possible for the Pendulum of a 26,7
 Clock to measure Time by its least Vi- 2,2
 brations, and that Pendulum were after- 6322,7
 wards made to swing in an Arc of 120 6333,0
 Degrees, it would lose 1 Hour, 45 Mi-
 nutes, 23 Seconds every Day.

N. B. That the second Term of the Series in
 an Arc of 15 Degrees does not amount to a tenth
 Part of a Second, nor the third in an Arc of 47
 Degrees, nor the fourth in an Arc of 81 De-
 grees.

PROBLEM II.

To measure the Perimeter of an ELLIPSE. Fig. 48.

Let $A B a b$ be an *Ellipse* similar to that which
 is proposed, wherein let $A C$ Half the transverse
 Axe be equal to 1, $F C$ Half the Distance of the
 Foci equal to m , $M P$ an Ordinate to the Axe
 equal to y , and $P C$ its Distance from the Center
 equal to x : Then will the Square of Half the lesser
 Axe $B C$ be equal to $1 - m^2$; and from the Na-
 ture of the *Ellipse* we shall have $y^2 = 1 - m^2 \times 1 - x^2$;
 from whence may easily be deduced this fluxional

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Equation,

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Equation, viz. $\sqrt{\dot{x}^2 + y^2}$, or the Fluxion of the elliptical Arc B M, $= \sqrt{1 - m^2 x^2} \times \frac{\dot{x}}{\sqrt{1 - x x}}$. Throw

$\sqrt{1 - m^2 x^2}$ into an infinite Series, and let $\frac{\dot{x}}{\sqrt{1 - x x}} = \dot{P}$,

$x^2 \dot{P} = \dot{Q}$, $x^2 \dot{Q} = \dot{R}$, &c. as in the *Scholium* to the the second *Lemma*, and we shall have the elliptical Arc B M $= P - \frac{1}{2} m^2 Q - \frac{1}{2} m^4 R - \frac{1}{8} m^6 S - \frac{1}{12} m^8 T$, &c. in which Series the first Term P, or the

Fluent of $\frac{\dot{x}}{\sqrt{1 - x x}}$ is a circular Arc, whose Radius is 1, and whose right Sine is x , as is evident both from its own Nature, and from that of the Series.

Let the Point M now coincide with A, and the circular Arc P will become a quadrantal Arc, and the Quantity $\sqrt{1 - x x}$ now vanishing, it is evident from the *Scholium* abovementioned, that the Quantities P, Q, R, S, T, &c. will now be derived from one another by a continual Multiplication of these, viz. P, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{8}$, $\frac{7}{8}$, &c. But the Terms of the other Series $1 - \frac{1}{2} m^2 - \frac{1}{2} m^4 - \frac{1}{8} m^6 - \frac{1}{12} m^8$, &c. arise from a continual Multiplication of these, viz. 1, $-\frac{1}{2} m^2$, $+\frac{1}{4} m^2$, $+\frac{3}{8} m^2$, $+\frac{5}{8} m^2$, $+\frac{7}{8} m^2$, &c. Therefore by the first *Lemma*, the elliptical Arc B A will be equal to a Series formed by a continual Multiplication of these Quantities, viz. P, $-\frac{1}{4} m^2$, $+\frac{3}{12} m^2$, $+\frac{1}{3} m^2$, $+\frac{5}{24} m^2$, $+\frac{6}{24} m^2$, &c. This is upon a Supposition that the transverse Axe of the *Ellipse* is equal

equal to 2; but if that Axe be of any other Length and bears the same Proportion to the Distance of the Foci as 1 to m ; then making P equal to the Circumference of a Circle whose Diameter is the same with the Axe of the *Ellipse*, the whole Perimeter of the Ellipse expressed in *Newton's Way* will be $P - \frac{1}{4}m^2A + \frac{3}{128}m^2B + \frac{1}{32}m^2C + \frac{3}{256}m^2D + \frac{63}{16384}m^2E$, &c. which Series may also be expressed thus, $P + \frac{0-1}{4}m^2A + \frac{4-1}{16}m^2B + \frac{16-1}{36}m^2C + \frac{36-1}{64}m^2D + \frac{64-1}{100}m^2E$, &c.

In Partem posteriorem Propositionis quartae
LOGOMETRIAE. Fig. 49.

Sit altera Asymptotos $C\beta$, et producat $BA\beta$, producat etiam $ZQ\varpi\xi$ occurrens reliquo Hyperbolae Cruri in ϖ , et Asymptoto $C\beta$ (quantum opus productae) in ξ ; et per Articulum 93. Conicorum *Hospitalii*, erit Rectangulum $P\xi \times PZ$ aequale Rectangulo $BA \times A\beta$; sed per Articulos 83, 118, est $Q\varpi = QP$, et per Articulum 95, $\varpi\xi = PZ$: Unde et $Q\xi$, QZ aequantur, et $P\xi$, aequalis est ipsi $\overline{QZ + QP}$, et PZ ipsi $\overline{QZ - QP}$, et Rectangulum $P\xi \times PZ$ aequatur Rectangulo $\overline{QZ + QP} \times \overline{QZ - QP}$; aequatur etiam $A\beta$ ipsi AB per Art. 109. ergo Rectangulum $\overline{QZ + QP} \times \overline{QZ - QP}$ aequatur Quadrato ipsius AB , ergo Rationes $\overline{QZ + QP}$ ad AB , AB ad $\overline{QZ - QP}$,
P 3 et

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et subduplicata Ratio $\overline{QZ+QP}$ ad $\overline{QZ-QP}$ eadem sunt. Porro, est AB ad QZ ut CA ad CQ , et AD ad QP ut CA ad CQ ; ergo est AB ad QZ ut AD ad QP , et permutando AB ad AD ut QZ ad QP , et permiscendo $AB+AD$ ad $AB-AD$ ut $QZ+QP$ ad $QZ-QP$; ergo Rationes omnes $QZ+QP$ ad AB , AB ad $QZ-QP$, subduplicata Ratio $QZ+QP$ ad $QZ-QP$, et subduplicata Ratio $AB+AD$ ad $AB-AD$ aequantur inter se: Hasce Rationes *priores* nominabimus.

Rursus, Rectangulum $\overline{RP+RX} \times \overline{RP-RX}$ aequale esse Quadrato ex CA ex iisdem Articulis constat, quibus ostensum est Rectangulum $\overline{QZ+QP} \times \overline{QZ-QP}$ aequale esse Quadrato ipsius AB ; quare est $RP+RX$ ad AC ut AC ad $RP-RX$, quare Rationes omnes $RP+RX$ ad AC , AC ad $RP-RX$, et subduplicata Ratio $RP+RX$ ad $RP-RX$ aequantur inter se; hasce *posteriores* nominabimus. Restat ut ostendamus Rationem AE ad PF aequalem esse Rationibus singulis tum prioribus, tum posterioribus, et propterea Rationes singulas tum priores tum posteriores aequales esse inter se; id vero sic evincitur: Triangula AEB , PFZ , similia sunt propter Parallelismum laterum, adeoque est AE ad PF ut AB ad PZ , vel AB ad $QZ-QP$; sed Ratio AB ad $QZ-QP$ una est ex Rationibus prioribus, ergo Rationes omnes priores aequales sunt Rationi ipsius AE ad PF . Rursus Triangula AEC , PFX

P F X etiam similia sunt propter Parallelismum laterum, ergo est A E ad P F ut A C ad P X; sed Ratio A C ad P X una est ex Rationibus posterioribus, ergo Rationes omnes posteriores aequales sunt Rationi A E ad P F, et Rationes omnes tum priores, tum posteriores, aequantur inter se. Haec de Rationibus.

Ostendendum secundo Sectorem Hyperbolicum C A P aequalem esse Spatio mixtilineo A E F P, quod sic demonstrari potest. Triangulum C A E aequatur Triangulo C P F, per Art. 99, priori Triangulo C A E, addatur Triangulum A E B, et fiet Triangulum C A B, subducatur Triangulum C D B et relinquetur Triangulum C A D, addatur Trilineum D A P, et conficietur Sector Hyperbolicus C A P: Alteri Triangulo C P F subducatur Triangulum C D B et manebit Trapezium D B F P, adjiciatur Trilineum D A P et fiet Figura mixtilinea A B F P; quae adsciscens Triangulum A E B, evadet Figura mixtilinea A E F P. Ergo Sector Hyperbolicus C A P aequatur Spatio A E F P cum ex aequalibus Elementis tum additiis tum subductitiis componatur utraque Figura.

Postremò ostendendum Sectorem Hyperbolicum C A P mensuram esse Rationum aequalium praefinitarum ad Modulum aequalem Triangulo C A B; id vero constat ex priori Parte hujus Prop. Nam Spatium A E F P est mensura Rationis inter A E et P F ad Modulum aequalem Parallelogrammo, quod duplum est Trianguli C A E. Est autem Spatium A E F P aequale Sectori C A P, et Triangulum

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gulum CAB duplum est Trianguli CAE , propterea quod Longitudo CB dupla est Longitudinis CE , per Art. 107. Quare Sector Hyperbolicus CAP mensura est Rationis inter AE et PF , &c. ad Modulum aequalem Triangulo CAB . Q.E.D.

In Scholium ad Propositionem Quartam
LOGOMETRIAE. Fig. 50.

LEMMA.

Si in dato aliquo Systemate sit s Mensura Rationis alicujus ad Modulum S , fueritque in alio quovis Systemate t ad T ut s ad S , erit t Mensura Rationis ejusdem ad Modulum T . Nam in omni Systemate Modulus et Mensura Rationis ejusdem sunt proportionales. Hoc praemisso, pergo ad alia quae hoc in loco considerata veniunt, et sic se habent. Solverat *Newtonus* Problema de Descensu Gravium, quibus resistitur in duplicata ratione Velocitatis per Constructionem Hyperbolae, cujus vis aequilaterae AT , Centro D et vertice Principali A descriptae, et capiendo Sectorem DAT ad dimidium Trianguli DAC ut t ad T : Verum cum istiusmodi Sector Hyperbolicus desumi nequit nisi prius habeatur Punctum P , per quod Semidiameter altera DT transire debet, ut abscindat aream propositam, cumque post inventum Punctum P , nihil opus est, ut Problema ulterius referatur ad Hyperbolam, itaque *Cotesius* Constructionem *Newtonianam* de Hyperbolis ad simplicem Lineam rectam more suo transtulit in hunc modum.

Area

Area Sectoris DAT Mensura est rationis sub-
 duplicatae inter $AC + AP$ et $AC - AP$ ad Mo-
 dulum aequalem Triangulo DAC , vel est Mensura
 Rationis illius Integrae ad Modulum $\frac{1}{2} DAC$;
 itaque cum sit ex Hyp. t ad T ut Mensura Ra-
 tionis inter $AC + AP$ et $AC - AP$ ad Mensurae
 Modulum $\frac{1}{2} DAC$, erit per *Lemma* praemissum, t
 Mensura Rationis inter $AC + AP$ et $AC - AP$
 ad Modulum T : Fiat ergo ut T ad t ita Modulus
Briggianus ad quartum, sitque quartus ille Lo-
 garithmus Fractionis $\frac{r}{s}$; et erit $AC + AP$ ad
 $AC - AP$ ut R ad S , et prodibit $AP = AC \times$
 $\frac{R - S}{R + S}$. Quare si exponatur recta quaevis AC et
 ita abscindatur longitudo AP , ut sit ad AC ut
 $R - S$ ad $R + S$, recta illa AC ita secabitur in P ,
 ut t sit Mensura Rationis inter $AC + AP$ et
 $AC - AP$ ad Modulum T . Invento jam hoc
 modo Puncto P , reliqua facili Negotio eruuntur,
 ex Propositionibus octava et nona Libri secundi
Principiorum; erit enim v ad U ut AP ad AC ,
 item r ad R ut $\overline{AP}^2$ ad $\overline{AC}^2$; vel si capiantur
 AC , AP , AK continuae Proportionales, erit r
 ad R ut AK ad AC ; denique Spatium s erit
 Mensura Rationis inter AC et CK , nam ex Pro-
 positionibus modo citatis, est s ad S ut Area altera
 Hyperbolica $ABNK$ ad Rectangulum $CK \times KN$,
 b, e , ut Mensura Rationis inter CA et CK , ad
 Modulum Systematis.

Haec

Haec utique est Constructio *Cotesiana* simplici Linea recta ut supra diximus exposita; sed et illud tamen obiter notandum, si harum Demonstrationem aggressus esset *Cotesius*, id ex notis Geometriae Vulgaris principiis minime praestari posse, nisi adhibitis Hyperbolis *Newtonianis*; unde satis liquet Constructionem hanc *Cotesianam* nihil aliud esse quam *Newtonianam* illam ad praxin deductam; sicut et ipse *Newtonus* eandem ad Numeros reduxit, posteaquam eo Loci perventum esset, ut solveretur Problema ex Phaenomenis. *Vide Prop. quadragessimam Libri secundi Principiorum.*

In dicti Scholii partem alteram, de RESISTENTIA in ASCENSU.

LEMMA.

Arcus Circuli 57,3 Graduum fere equalis est Radio, nam Circumferentia Circuli est ad Semidiametrum suam ut 355 ad $\frac{113}{2}$, vel ut 710 ad 113, vel ut 360 ad 57,3 *feré*.

COROLLARIUM.

Hinc si detur Numerus Graduum in Arcu quolibet, invenietur Longitudo Arcus in partibus Radii, dicendo ut 57,3 ad Numerum Graduum datum, ita Longitudo Radii ad Longitudinem Arcus: Et *è converso*, si detur Arcus Longitudo et quaerantur Gradus, dic ut Longitudo Radii ad Longitudinem Arcus datam, ita 57,3 ad Numerum Graduum quaesitum.

Caetera

Caetera quod attinet animadvertendum, Primo Si detur v Velocitas prima Corporis ascendentis, et quaeratur tum resistentia prima r , tum tempus t , et Spatium s totius Ascensus, acceptâ utcumque Longitudine EG , Fig. 51, ut et huic aequali et perpendiculari GO , Capiatur GF vel Gf ad GE ut v ad V , sitque Gb ipsis GE , GF tertia proportionalis, sed ad partes contrarias sita; et erit r ad R ut Gb ad GE . Jungantur OF , Of et ex datis OG , GF , quaeratur Angulus FOG , ut et hujus duplus FOf ; et invenietur Tempus t , dicendo ut 57,3 ad Angulum FOf , ita T ad t : Invenietur denique Spatium s , si fiat ut Modulus *Briggianus* ad Logarithmum *Briggianum* Fractionis $\frac{Eb}{EG}$, ita S ad s . Secundo, Si detur r et quaerantur reliqua, fiat Gb ad GE ut r ad R , et habebitur Punctum b , Capiatur GF vel Gf media proportionalis inter Gb et GE , et habebitur Punctum F ; quibus datis reliqua elicientur ut prius. Tertio, Si detur solum Tempus t et desiderentur reliqua; fiat ut T ad t , ita 57,3 ad Angulum FOf , et hujus dimidium FOG etiam innotescet; dato autem Angulo FOG et Radio OG , inveniatur Tangens GF , et dabitur Punctum F , et reliqua omnia ut supra.

Postremo, Si detur solum Spatium s , fiat ut S ad s ita Modulus *Briggianus* ad Logarithmum Fractionis $\frac{R}{S}$, et erit Eb ad GE ut R ad S unde dabitur Punctum b , et reliqua omnia.

COROLLARIUM.

Ex jam Demonstratis consequens est, si Corpus e quiete demittatur, Descensurum illud in Aeternum et per Spatium infinitum, prius quam possit acquirere Velocitatem, Velocitati finitae V aequalem; et si sursum projiciatur cum Velocitate infinita, Ascensurum illud per Spatium infinitum prius quam ad Altitudinem Summam pervenire possit, cum tamen Tempus totius Ascensus finitum sit, et ad Tempus T ut Peripheria Circuli ad Diametrum.

In partem posteriorem Scholii ad Prop. quintam
LOGOMETRIAE.

LEMMA.

Si duae Quantitates Homogeneae et indeterminatae A et B ita inter se comparatae fuerint ut sit A semper ut $A-B$, erit etiam A semper ut B ; mutetur enim A in a , et simul B in b , et cum sit *ex Hyp.* A ut $A-B$, *b. e.* cum sit A ad $A-B$, semper in data Ratione, erit A ad $A-B$ ut a ad $a-b$, et Dividendo A ad B ut a ad b ; *b. e.* erit A ut B . Q. E. D.

In parte hujus *Scholii* posteriore, Solvitur *Problema* de Densitate Aëris ad quamvis Altitudinem invenienda, ubi Vis Gravitatis diminuitur in duplicata Ratione auctae Altitudinis a Centro Terrae; et certè Solutio hic allata perquam simplex est et elegans; haud tamen inducar ut credam demonstrationem hujusce Solutionis aut adeo concinnatam

natam esse, aut Tyronum mentibus accommodatam, ut ulteriori illustratione non sit opus; hanc itaque pro Modulo meo aliquanto dilucidatiorem dare conabor, et summa qua possum brevitate expediam.

Concipe ergo Columnam Aeris Cylindricam a Superficie Telluris ad Summam usque Atmosphaeram pertinentem, et, servata Auctoris constructione, Fig. 52. cum sit SF ad SA ut SA ad Sf erit $\overline{SA^2}$ aequali Rectangulo $SF \times Sf$; sed et idem $\overline{SA^2}$ atque eadem Ratione etiam aequatur Rectangulo $SM \times Sm$: Quare Rectangula $SF \times Sf$ et $SM \times Sm$, aequantur inter se, estque SM ad SF ut Sf ad Sm ; et *dividendo*, FM ad SF ut fm ad Sm vel Sf ; unde *vicissim* est FM ad fm ut SF ad Sf ; sed propter SF , SA , Sf , continue proportionales, est SF ad Sf ut $\overline{SF^2}$ ad $\overline{SA^2}$, quare FM est ad fm ut $\overline{SF^2}$ ad $\overline{SA^2}$, estque fm aequalis $\frac{SA^2}{\overline{SF^2}} \times FM$; unde cum detur $\overline{SA^2}$ est fm ut $\frac{FM}{\overline{SF^2}}$, et $fm \times fg$, sive Areola $fgnm$, ut $\frac{FM \times fg}{\overline{SF^2}}$: Sed $\frac{FM \times fg}{\overline{SF^2}}$ est ut moles Aeris inter F et M , atque ut Densitas ejusdem et Vis Gravitatis in Loco F ; quae simul sunt ut pondus Aeris inter F et M ; quare pondus Aeris inter F et M est ut Area $fgnm$. Rursus, densitas et pressio Aeris in Loco F exponitur per Ordinatam fg , et in Loco M , per Ordinatam mn , ergo differentia pressionum in Locis F et M , exponenda est per differentiam

Ordi-

Ordinatarum nempe per $fg - mn$; sed haec differentia nihil aliud est quam pondus Aëris inter F et M; quare pondus Aëris inter F et M est ut $fg - mn$: Sed ostensum est et hoc pondus esse ut Area $fgnm$, quare Area $fgnm$ est ut $fg - mn$. Haec ita sunt universaliter: Quod si detur Distantia fm , erit Area $fgnm$ ut fg , quare in hoc Casu erit fg ut $fg - mn$, et proinde etiam fg ad mn , semper in data Ratione; sicut in praecedenti Lemmate. Exponat jam data illa distantia fm datam hanc Rationem inter fg et mn , et Summa omnium fm exponet similem Summam Rationum omnium $\frac{fg}{mn}$; eâ est itaque indole haec Curva ut binarum quarumlibet ordinatarum intervallum. Mensura sit Rationis inter ipsas Ordinatas, quae est notissima proprietas *Logistica*: Est ergo Curva Bg Logistica. Dico et eandem esse cum priore BG, sed inverso situ; quod sic ostendo. Accedere intelligantur Puncta F ad Punctum A in utraque Figura, ita tamen ut Longitudines AF, AF, semper sint invicem aequales, et coeuntibus F et A fiet Lineola Af aequalis ipsis AF, etenim cum sit SF ad SA ut SA ad Sf, et *dividendo* AF ad SA ut Af ad Sf, erit *vicissim* AF ad Af ut SA ad Sf; *b. e.* ultimo in Ratione aequalitatis, unde cum Subtangens Curvae BG sit $\frac{AB \times AF}{AB - FG}$, et Subtangens Curvae Bg sit $\frac{AB \times Af}{AB - fg}$, erit Sub-

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tangens Curvae BG ad Subtangentem Curvae Bg

ut $\frac{I}{AB-FG}$ ad $\frac{I}{AB-Fg}$.

Porro, pressio Aëris et Vis Gravitatis in Loco A eadem ponuntur in utraque Hyp. et coëuntibus A et F Vis Gravitatis inter A et F in secunda Hypothesi pro uniformi tuto haberi potest; quare pondus Aëris inter A et F, idem est in utraque Hypothesi; unde et horum ponderum, exponentes, numerum AB-FG in prima Figura et AB-fg in secunda sunt aequales; ergo Curvarum Subtangentes etiam aequales sunt; quare Figura Bg, eadem omnino est cum Figura BG; harum enim Constructiones ex solis Subtangentibus pendere, constat ex praecedentibus. Q. E. D.

Theoremata duo pro comparandis Areis curvilineis formae simplicioris.

Sit P Area Curvae cujus Abscissa est z, et ordinatim, applicata $z^{\theta-1} \times e^{\pm f z^{\lambda-1}}$, sit etiam T vel T' Area Curvae cujus eadem est Abscissa z, ordinatim vero Applicata $z^{\theta \pm \sigma-1} \times e^{\pm f z^{\lambda-1}}$, posito σ pro Numero quolibet integro; quaeritur Relatio inter P et T, vel inter P et T'.

SOLUTIO.

$$\text{Fiat } \frac{I}{\theta} = p, \frac{-pf}{e} \times \frac{\theta+\lambda}{\theta+1} = q, \frac{-qf}{e} \times \frac{\theta+\lambda+1}{\theta+2} = r, \\ \frac{-rf}{e} \times \frac{\theta+\lambda+2}{\theta+3} = s, \text{ et } \frac{-sf}{e} \times \frac{\theta+\lambda+3}{\theta+4} = t; \text{ si modo}$$

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modo fit s ultimus Terminus Seriei p, q, r, s , quorum tot sumendi sunt quot sunt Unitates in σ , fiat etiam Regressus a Termino p , ad Terminos p', q', r', s', t' , sed mutatis omnium Signis; *b. e.*

$$\text{cum sit } \frac{-se}{f} \times \frac{\theta+3}{\theta+\lambda+2} = r, \quad \frac{-re}{f} \times \frac{\theta+2}{\theta+\lambda+1} = q,$$

$$\frac{-qe}{f} \times \frac{\theta+1}{\theta+\lambda} = p, \quad \text{fiat etiam } \frac{pe}{f} \times \frac{\theta}{\theta+\lambda-1} = p',$$

$$\frac{-p'e}{f} \times \frac{\theta-1}{\theta+\lambda-2} = q', \quad \frac{-q'e}{f} \times \frac{\theta-2}{\theta+\lambda-3} = r',$$

$$\frac{-r'e}{f} \times \frac{\theta-3}{\theta+\lambda-4} = s', \quad \text{et } -s' \times \theta-4 = t', \text{ si modo}$$

fit s' ultimus Terminus Seriei p', q', r', s' , quorum etiam tot sumendi sunt quot sunt unitates in σ .

$$\text{His positis erit } P = \overline{p z^{\theta n} + q z^{\theta n+1} + r z^{\theta n+2} + s z^{\theta n+3}} \times \frac{e+f z^{\lambda}}{\eta e} + t T, \text{ si modo habeatur } +\sigma \text{ in Valore}$$

ordinatae, sin habeatur $-\sigma$, erit $P = \overline{p' z^{\theta n-n} +}$

$$\overline{q' z^{\theta n-2n} + r' z^{\theta n-3n} + s' z^{\theta n-4n}} \times \frac{e+f z^{\lambda}}{\eta e} + t' T'.$$

N. B. Si fiat $-\theta - \lambda + 1 = +\varpi$ erit ordinata $z^{\theta n-1} \times e + f z^{\lambda-1} = z^{-\varpi-1} \times f + e z^{-\lambda-1}$.

Concinnari etiam potest Theorema secundum in modum sequentem; fiat $\frac{1}{\theta+\lambda-1} = p', \quad \frac{-p'e}{f}$

$$\times \frac{\theta-1}{\theta+\lambda-2} = q', \quad \frac{-q'e}{f} \times \frac{\theta-2}{\theta+\lambda-3} = r', \quad \frac{-r'e}{f} \times$$

$$\frac{\theta-3}{\theta+\lambda-4} = s', \quad \frac{-s'e}{f} \times \theta-4 = t', \text{ si modo fit } s' \text{ ul-}$$

timus Terminus Seriei p', q', r', s' , quorum tot sumendi

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sumendi sunt quot sunt Unitates in σ , et erit

$$P = \overline{p'z^{\theta\eta} + q'z^{\theta\eta-1} + r'z^{\theta\eta-2} + s'z^{\theta\eta-3}} \times \frac{e+fz^{\eta\lambda}}{\eta f z^{\eta}} + t' T'.$$

Demonstratur Theorema primum per Aequationes a nobis alibi traditas et demonstratas; nam si scribantur Q, R, S, pro Areis intermediis inter P et T, et y pro $\overline{e+fz^{\eta\lambda}}$, erit primo $\theta \times e P + \overline{\theta + \lambda} \times f Q = z^{\theta\eta} \times \frac{y}{\eta}$, secundo $\overline{\theta + 1} \times e Q + \overline{\theta + \lambda + 1} \times f R = z^{\theta\eta+1} \times \frac{y}{\eta}$, tertio $\overline{\theta + 2} \times e R + \overline{\theta + \lambda + 2} \times f S = z^{\theta\eta+2} \times \frac{y}{\eta}$, quarto $\overline{\theta + 3} \times e S + \overline{\theta + \lambda + 3} \times f T = z^{\theta\eta+3} \times \frac{y}{\eta}$: harum Aequationum ope exterminentur Fluentes intermediae Q, R, S, et elicietur Aequatio exhibens Relationem inter P et T ut supra. Porro demonstrari potest Theorema secundum, vel ex priore vel ad modum prioris.

COROL. I.

Si Series alterutra abrumpatur propter Terminos evanescentes vel infinitos, dabitur Valor Quantitatis alterius P vel T, vel Quantitatis alterius P vel T' absque ulla relatione ad alteram, praesertim post divisionem Aequationis totius per Quantitatem infinitam si qua in Aequatione existit.

COROL. II.

Quod si Quantitas T, vel T' abeat in infinitum, prodibit Series *Newtoniana* pro inveniendо Valore Quantitatis P, accurate quidem, si abrumpatur; vel per Approximationem, si procedat in infinitum.

Q

A P R O-

A PROBLEM.

To measure the Quantity of Surface generated by an Arc of the Rectangular HYPERBOLA revolving about one of its Asymptotes. See Harmon. Menfur. p. 94.

SOLUTION.

Let m^2 be the Power of the Hyperbola referred by perpendicular Ordinates to either of its Asymptotes, and let y represent indifferently any one of these Ordinates whose Absciss or Distance from the Center let be x ; and lastly, let the Diameter of a Circle be to its Periphery as 1 to p , then will

$2py \times \sqrt{\dot{x}^2 + \dot{y}^2}$ be the Fluxion of the Surface sought; or, dropping p till the Conclusion, the Fluxion will be $2y \times \sqrt{\dot{x}^2 + \dot{y}^2}$, out of which either $\dot{x}$ or $\dot{y}$ must be expunged by the Nature of the Hyperbola. Let $\dot{x}$ be expunged thus, According

to the Nature of the Hyperbola, $x = \frac{m^2}{y}$, there-

fore $\dot{x}^2 = \dot{y}^2 \times \frac{m^4}{y^4}$ and $\dot{x}^2 + \dot{y}^2 = \dot{y}^2 \times \frac{m^4 + y^4}{y^4}$, there-

fore $\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{y}}{y^2} \times \sqrt{m^4 + y^4}$, I say $-\dot{y}$, &c.

because y flows contrarily to our Surface, whose Fluxion was put affirmative: Therefore $2y \times \sqrt{\dot{x}^2 + \dot{y}^2} = -2\dot{y} \times \sqrt{m^4 + y^4}$. But this is a Fluxion of the *third Form*, where $\theta = 0$, $\eta = 4$, $d = -2$, $z = y$, $e = m^4$, $f = 1$, $P = \sqrt{m^4 + y^4}$, $R = m^2$,
T =

$T = \sqrt{m^4 + y^4}$, $S = y^2$; therefore $\frac{2}{y} dP - \frac{2}{y} dR \bigg| \frac{R + T}{S}$, or the whole Fluent sought, will be $-\sqrt{m^4 + y^4} + m^2 \bigg| \frac{m^2 + \sqrt{m^4 + y^4}}{y^2}$: But from the Nature of the Hyperbola, $m^2 = yx$, and $m^4 = y^2 x^2$, and $m^4 + y^4 = y^2 x^2 + y^2$, and $\sqrt{m^4 + y^4} = y \times \sqrt{x^2 + y^2}$; substitute therefore $y \times \sqrt{x^2 + y^2}$ instead of $\sqrt{m^4 + y^4}$ in the foregoing Fluent, as also xy instead of m^2 in the Ratio $\frac{m^2 + \sqrt{m^4 + y^4}}{y^2}$, and the Fluent will stand thus, $-y \times \sqrt{x^2 + y^2} + m^2 \bigg| \frac{xy + y \times \sqrt{x^2 + y^2}}{y^2}$
 $= -y \times \sqrt{x^2 + y^2} + m^2 \bigg| \frac{x + \sqrt{x^2 + y^2}}{y}$.

Now to construct this Fluent, let A be the Center, and AB the Asymptote referred to (See Cotes, Page 94.) let BC be a fixt Ordinate, and DE a moveable one, comprehending between them the generating Arc CE, and the Line AE when drawn will be $\sqrt{x^2 + y^2}$, since AD = x and DE = y; therefore $y \times \sqrt{x^2 + y^2}$ is the Rectangle AED, or AE x ED; moreover the Modulus m^2 will in this Case be the Rectangle ABC, therefore according to this Construction, the general Fluent will be $-AED + ABC \bigg| \frac{DA + AE}{DE}$. But when D coincided with B, this Fluent was $-ACB + ABC \bigg| \frac{BA + AC}{BC}$; therefore so much of the

Fluent as makes for our Purpose will be $ACB -$

$$AED + ABC \left| \frac{DA + AE}{DE} - ABC \right| \frac{BA + AC}{BC}.$$

Now, for a more convenient Subtraction, let the Rectangles AED and ACB be reduced to a

common Base, as well as the Ratio's $\frac{DA + AE}{DE}$

and $\frac{BA + AC}{BC}$ to one common Consequent, thus;

Let the Line AE cut BC in F , and we shall have, from the Nature of the Hyperbola, BC to DE as AD to AB , or as AE to AF ; therefore by multiplying Extremes and Means we have $AED = AF \times BC$; therefore $ACB - AED$, or the rational Part of the Fluent, $= AC - AF \times BC$.

Again, draw CG parallel to AE , and the Ratio $\frac{DA + AE}{DE}$ will be equal to the Ratio $\frac{BG + GC}{BC}$,

and the Excess of the Ratio $\frac{DA + AE}{DE}$, above the Ratio $\frac{BA + AC}{BC}$ will now be the Excess of

the Ratio $\frac{BG + GC}{BC}$ above the Ratio $\frac{BA + AC}{BC}$,

which Excess is equal to the Ratio $\frac{BG + GC}{BA + AC}$;

and the Fluent will stand thus, $AC - AF \times BC$

$+ ABC \left| \frac{BG + GC}{BA + AC} \right|$. Make $AC - AF + AB \left|$

$\frac{BG + GC}{BA + AC} = AH$; and then the Fluent (restor-

ing p , that was set aside) will be $p \times BC \times AH$,
 or $p \times \overline{BC}^2 \times \frac{AH}{BC}$; but $p \times \overline{BC}^2$ is the Area of a
 Circle whose Semi-diameter is BC , therefore the
 Surface sought is to the Area of a Circle whose
 Semi-diameter is BC , as AH to BC . Q. E. D.

A THEOREM. Fig. 54.

*If the Space between any Curve and its Asymptote be
 supposed to turn round that Asymptote, so as to
 generate a Tapering Solid of an infinite Length;
 I say, that the Surface of this Solid will be finite or
 infinite in Quantity, according as the Quantity of
 the Space, that generates this Solid, is so.*

Let the streight Line $ACDF$ be an Asymptote
 to the Curve BE , and let the Space betwixt them
 be terminated at one End by the Line AB per-
 pendicular to the Asymptote AF , let DE and de
 be any two neighbouring Ordinates parallel to
 AB , and let BC and EF be Tangents to the
 Curve in the Points B and E ; lastly, let g be the
 Circumference of a Circle whose Diameter is DE :
 Then if we suppose the Ordinate de to approach
 nearer and nearer to the Ordinate DE , and at last
 to coincide with it, the Fluxion of the Surface at
 E will be to a correspondent Fluxion of the Space
 $ADEB$ in the ultimate Ratio of $g \times Ee$ to
 $DE \times Dd$; but this Ratio is compounded of two
 others, viz. that of g to DE and that of Ee to
 Dd ; the first of these Ratio's, viz. that of g

230 *When finite, when infinite.*

to DE will always be finite and the same, being the Ratio of the Circumference of a Circle to the Radius, the other component Ratio, *viz.* that of Ee to Dd tends to a Ratio of Equality ; because the nearer the Curve approaches to its Asymptote, the nearer will the Lines Ee and Dd approach towards a State of Parallelism and Equality : Therefore the Ratio of Ee to Dd or of EF to DF will always be less than that of BC to AC ; but the Ratio of BC to AC will always be finite, when the Ordinate AB cuts and does not touch the Curve ; therefore, *a fortiori*, the Ratio of Ee to Dd must be finite ; therefore every Fluxion of the Surface will be to a like Fluxion of the Asymptotic Space in a finite Ratio, since this Ratio will always be the Sum of two others which are both finite ; therefore the whole Surface will be to the whole Space in a finite Ratio ; therefore the Surface must be finite or infinite in Quantity, according as the Quantity of the Space that generates the Solid, is so. Q. E. D.

V. Cl.

Plate IX.

Fig. 47.

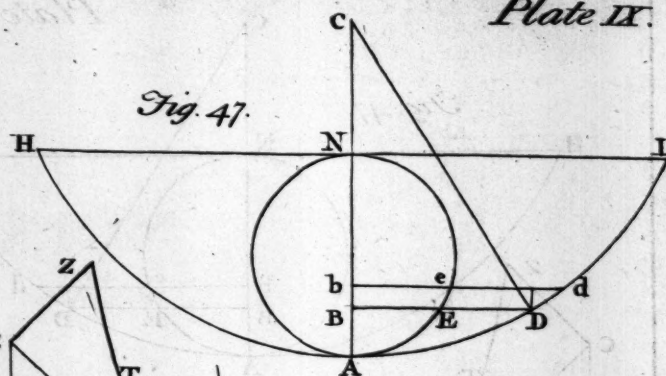


Fig. 48.

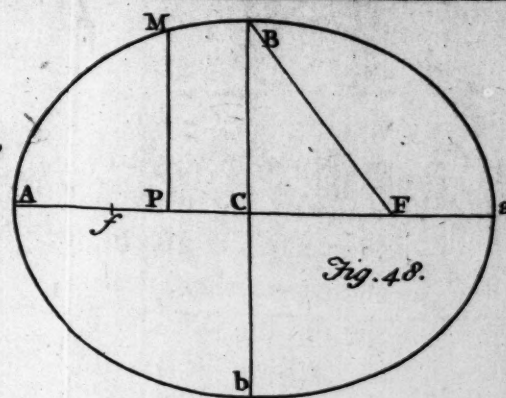


Fig. 50.

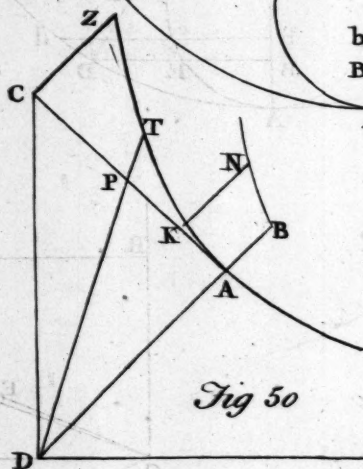


Fig. 49.

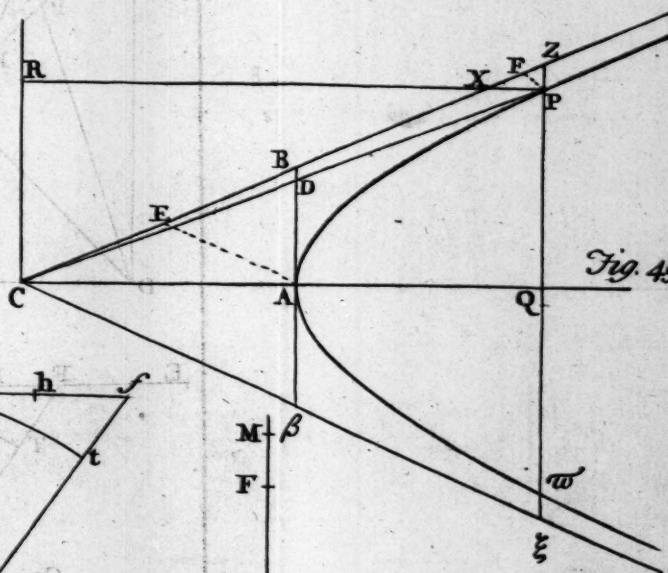


Fig. 51.

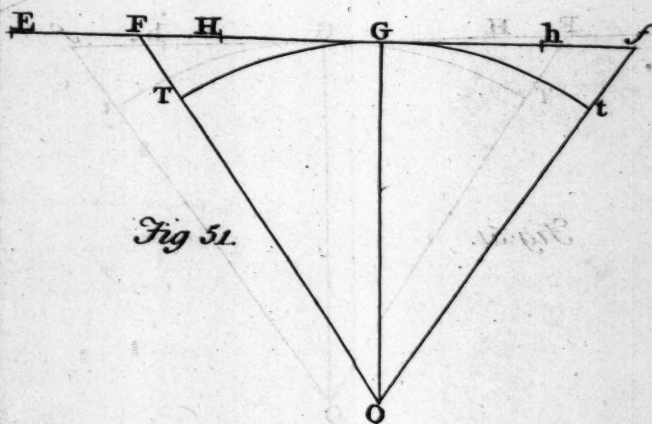


Fig. 52.

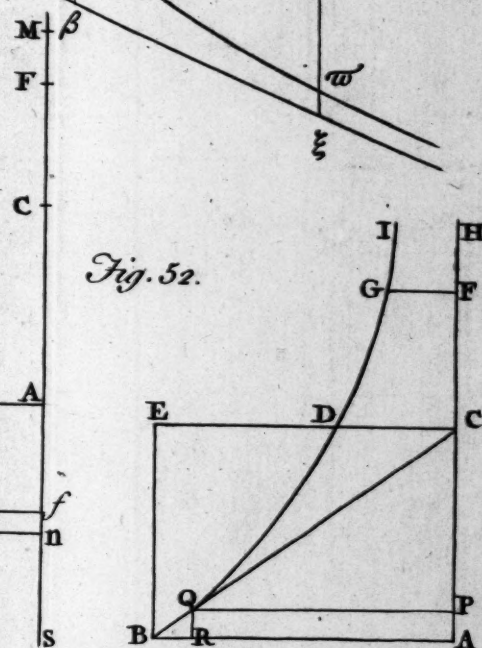
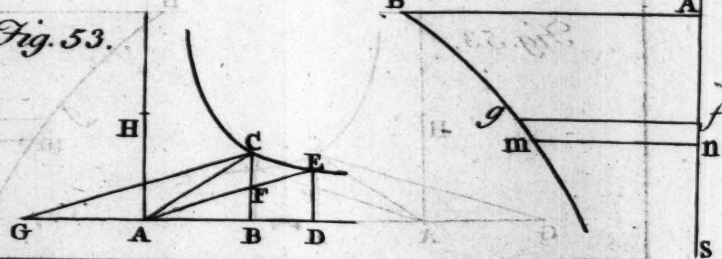


Fig. 53.



V. Cl.

NICOLAI SAUNDERSON

I N

PHILOSOPHIAM NEUTONIANAM

S C H O L I A.

PROBLEMA *ad illustrandum* COROL. II. *ad*
Leges Motûs. Fig. 55.

EX-Punctis M et N in Plano quovis ad Horizontem normali suspendantur Corpora A et p, ita tamen ut Corpus A liberè dependeat, et Corpus p incumbat Plano dato p G; *quaeritur Ratio Ponderum A et p ita comparatorum ut se mutuò in Aequilibrio sustineant.* Oportet autem ut innotescat Ratio Distantiarum minimarum O K, o k, Filorum M A, N p a Centro O, et Ratio Rectarum N p, p H; positâ H N ad p G normali.

Filo N P supponatur Trochlea Z, dein tollatur Planum p G, ita ut Fili Pars N Z eandem servet Positionem quam prius; pars autem Z P liberè dependeat; et Corpus P jam toto suo Pondere tendet Filum N Z. Sit Corpus A 20 Librarum, sitque minima Distantia Centri O a Filo M A ad

Q 4 minimam

minimam Distantiam ejusdem Centri O a Filo NZ ut 3 et 4, et si Pondus P sustineat Pondus A in Aequilibrio, erit hujus Pondus 15 Librarum, propterea quod sit 20 ad 15 ut 4 ad 3. Restituatur jam Corpus P in Locum suum priorem, nimirum ut incumbat Plano p G, et Tensio Fili minuetur hâc Translatione in Ratione p H ad p N. Sit p H ad p N ut 6 ad 5, et Tensio Fili N p jam minuetur in hâc Ratione; ergo Pondus P jam non ampliùs in Aequilibrio sustinebit Pondus A, nisi tantò augeatur Pondus absolutum Corporis P, quantò diminuta erat Tensio Fili per Corporis p Translationem: Quare ut Corpus P sustineat Corpus A in Aequilibrio, oportet ut Pondus ejus absolutum 15 Librarum augeatur in Ratione 5 ad 6, vel in Ratione 15 ad 18. Quare si Pondus A sit 20 Librarum, oportet ut Pondus Corporis p Plano p G incumbentis sit 18 Librarum; hâc enim Ratione Corpora A et p in Aequilibrio consistent; et Ratio 20 ad 18 componitur ex Rationibus 20 ad 15 et 15 ad 18, hoc est, ex Rationibus 4 ad 3, et 5 ad 6.

Proprietates

Proprietates quaedam Centri GRAVITATIS.

THEOREMA. Fig. 56.

Si Corpora quocunque (utcunque sita inter se) A, B, C, Viribus quibuscunque de Locis suis A, B, C, simul impellantur, commune eorum Gravitatis Centrum perinde movebitur, tum quoad Motûs sui Determinationem, tum quoad Velocitatem, atque Corpus Unicum $A+B+C$ ex omnibus Compositum et in commune omnium Gravitatis Centro constitutum; si modo Corpus illud compositum iisdem quibus componentia, Viribus simul impressis, et similiter determinatis, impellatur.

Esto enim S commune omnium Gravitatis Centrum, et excluso Corpore A, sit T commune Gravitatis Centrum reliquorum B, et C; et Corporibus in Locis A, B, C, primò quiescentibus, ut et Centro S in Loco E; transferatur postea (reliquis quiescentibus) Corpus A in Locum *a*, atque adeo Centrum S de Loco E, in Locum quendam F, ubi rursus quiescant, et Punctum F erit in Recta T *a*, ex Natura Centri Gravitatis; erit etiam Recta E F, Rectae A *a* parallela; ex eo quod Rectae T A, T *a* similiter secantur in Punctis E et F; Denique cum sit ex Natura Centri Gravitatis, T E ad E A, ut A ad B+C; erit componendo T E ad T A vel E F ad A *a* ut A ad A+B+C. Transferatur jam Corpus B in Locum *b*, et simul Centrum S de Loco F in Locum G,
ubi

ubi rursus quiescant; et Recta FG parallela erit Rectae B *b*, eritque ad B *b* ut Corpus B ad summam Corporum A + B + C. Postremò transferatur Corpus C in Locum *c*, et simul Centrum S de Loco G in Locum H; et Recta GH parallela erit Rectae C *c*, eritque ad C *c* ut Corpus C ad summam Corporum A + B + C; Restituantur jam Corpora in Loca sua prima A, B, C, et Viribus quibuscunque de his Locis secundum directiones, et cum Velocitatibus A *a*, B *b*, C *c*, simul impellantur, nempe ut ad Loca *a*, *b*, *c*, eodem Tempore *t* simul appellant; et Corporum in his Locis simul existentium, commune Gravitatis Centrum, erit in Loco H ut supra definito.

Fingatur jam Corpus Unicum D quod aequale sit omnibus A, B, C, simul sumptis, et statuatur hujus Centrum in Loco E; et si huic Corpori (hoc est Centro ejus) imprimeretur sola vis motrix Corporis A, ob suppositam Virium Aequalitatem, et similem Determinationem, Corpus illud D moveretur in Recta EF cum Velocitate quae foret ad Velocitatem Corporis A, ut Corpus A ad Corpus D, vel ut EF ad A *a*; ergo Vis motrix Corporis A Corpori D impressa efficeret ut hocce percurreret Longitudinem EF, quo Tempore Corpus A percurrerat Longitudinem A *a* (id est Tempore *t*); et pari Ratione sola Vis motrix Corporis B ipsi D impressa, efficeret ut Corpus illud ex E decedens percurreret Longitudinem aequalem et parallelam ipsi FG eodem Tempore *t*; et sola vis motrix Corporis C impelleret Corpus D,

per

per Longitudinem aequalem et parallelam ipsi GH Tempore praefinito t : Quare Vires omnes simul et semel impressae, efficient ut Corpus D percurrat Longitudinem EH Tempore t , ita ut in fine Temporis illius reperiatur in H Loco communis Centri S; erat autem Tempus t pro Arbitrio assumptum: Quare si statuatur Centrum Corporis D in Centro communi totius Systematis A, B, C, et Corpus illud iisdem quibus Partes Systematis Viribus simul impressis et similiter applicatis, impellatur, erit hujus Centrum semper in Centro communi et idem erit motus utriusque. Q. E. D.

COROL. I.

Si Corpora A, B, C, moveantur uniformiter in directum, sive motus illi fiant in eodem Plano, sive non, Centrum Corporis D, et proinde etiam commune Gravitatis Centrum vel quiescet vel movebitur uniformiter in Directum.

COROL. II.

Longitudines EF, FG, GH, HE, sunt ut motus Corporum A, B, C, D: nam quo Tempore Corpus A percurrerat Longitudinem Aa, Corpus D percurrerat Longitudinem EH; ergo Velocitas Corporis D, est ad Velocitatem Corporis A ut EH ad Aa; et motus ad motum, ut $D \times EH$ ad $A \times Aa$; erat autem EF ad Aa ut A ad D, quare factum $D \times EF$ aequatur facto $A \times Aa$; et motus Corporis D, jam erit ad motum Corporis A ut $D \times EH$ ad $D \times EF$, seu ut EH ad EF; unde

unde si Longitudo EH referat motum Corporis D , Longitudines EF , FG , GH , pariter referent motus Corporum A , B , C .

COROL. III.

Hinc datis Motibus Corporum quocunque A , B , C , dabitur Motus communis omnium Centri Gravitatis; incipiendo nempe à Puncto quovis E et ducendo rectas EF , FG , GH , exhibentes Motuum illorum et Quantitates et determinationes; nam recta EH pariter exhibebit Motum totius Systematis, simpliciter cum Velocitate communis Centri Gravitatis lati, et quo Tempore Corpus A percurrit Longitudinem, quae sit ad EF ut D ad A , commune Centrum Gravitatis percurrat Longitudinem aequalem et parallelam ipsi EH .

COROL. IV.

Si Punctum ultimum H cum primo E coinciderit, Indicium erit commune Gravitatis Centrum quiescere.

COROL. V.

Quemadmodum Longitudo EH refert Motum Centri communis totius Systematis A , B , C , Longitudo EG referet Motum Centri communis partium A et B , et Longitudo FH Motum Centri communis partium B et C ; et si compleatur Parallelogrammum $FGHI$, Longitudo EI pariter exhibebit Motum Centri communis partium A et C . Haec ita sunt ex Hypothefi quod Longitudines

gitudines E F, F G, G H, referant Motus ipsorum A, B, C, respective.

COROL. VI.

Actiones extrinsecùs in partes Systematis A,B,C, non secus afficiunt Motum vel quietem Centri communis, quam similes et aequales Actiones in Centrum Corporis D exercitae afficiunt hujus Motum vel Quietem; neque refert quicquam in quas partes Systematis impetus fiant, modo virium Quantitates et Determinationes eadem maneant.

COROL. VII.

Quare vires aequales et contrariae, in partes Systematis simul impressae, nihil perturbabunt Motum vel Quietem Centri communis, siue vires illae in easdem siue in diversas partes Systematis imprimantur.

COROL. VIII.

Unde cum Actioni semper aequali et contraria sit Reactio, status Centri communis vel quiescendi vel movendi uniformiter in Directum ab Actionibus partium Systematis inter se nullatenus perturbabitur aut mutabitur.

COROL. IX.

Est itaque eadem lex Motus Corporum duorum vel plurium quae Corporis solitarii; siquidem Motus Systematis pariter atque Corporis solitarii ex Motu Centri Gravitatis aestimetur, perseverabit hoc in statu suo quiescendi vel movendi uniformiter in
Directum

Directum, nisi quatenus a Viribus extrinsecus impressis cogitur statum illum mutare.

His adjungere liceat (ne alias pereat) Solutionem non inveniutam sequentis Problematis, cujus quidem Demonstrationem ut satis obviam, vel saltem ex praecedentibus haud difficulter deducendam praetereo.

PROBLEMA.

Positis omnibus ut in praecedenti propositione, Corporibus A, B, C, in rectis A *a*, B *b*, C *c*, uniformiter delatis, imprimantur Vires novae, quarum et Determinationes et Quantitates exponantur per Rectas L *l*, M *m*, N *n*, ita ut Vis L *l* sit ad Vim insitam quo Corpus A prius movebatur ut Recta L *l* ad Rectam E F; et sic de caeteris, quaeruntur Motus resultantes?

SOLUTIO.

Constructâ ut in Corol. tertio superioris Propositionis, Figurâ E F G H sit P locus Puncti E; quo notato transferatur Figura tota, secundum Determinationem L *l*, ita ut singula ejus Puncta percurrant Longitudines aequales et parallelas Rectae L *l*, et constituta Figura in hoc situ, notetur Q locus Puncti F: ex hoc loco transferantur rursus Figura similiter secundum Determinationem M *m*; dein sistatur ut notetur R locus Puncti G; hinc denique transferatur Figura secundum Determinationem N *n*, et notetur S locus Puncti H, et jungantur postremo P Q, Q R, R S, S P; dico has
Rectas

Rectas rite exhibere et Determinationes et Quantitates Motuum resultantium : *b. e.* Corpus A post Actionem Vis L l movebitur secundum Determinationem P Q cum Motu et Velocitate quae erunt ad Motum et Velocitatem priorem ut P Q ad E F ; et sic de reliquis. Movebitur etiam commune omnium Gravitatis Centrum, jam secundum Determinationem P S ; et cum Velocitate quae erit ad Velocitatem priorem ut P S ad E H. Q. E. I.

COROL. I.

Si Impressiones omnes eandem habuerint Determinationem vel contrarias, Figura E F G H nunc prorsum, nunc retrorsum, secundum hasce determinationes mota inter easdem semper parallelas continebitur. Quod si Actiones omnes ex una Parte simul sumptae aequales fuerint Actionibus omnibus ex altera, Figura illa E F G H, utcunque hac et illac inter easdem semper parallelas mota, redibit tamen ultimo in pristinum situm et locum ; et Figura E Q R H, exhibebit Motus resultantes. Unde rursus facile colligitur commune Gravitatis Centrum totius Systematis, a Viribus aequalibus et contrariis simul impressis nihil omnino passurum quoad statum suum vel quiescendi vel movendi uniformiter in directum.

COROL. II.

Si Motus illi, quos Vires L l, M m, N n, simpliciter et seorsim generarent, sint ut Corpora A, B, C, respective, et simul aequales Motui H E, fiantque
in

in eandem Plagam, *b. e.* in Plagam HE, Punctum H dum Figura tota EFGH, transfertur secundum Determinationem Virium, locabitur semper in Recta HE, cadetque ultimo in Punctum P locum primum Puncti E. Unde consequens est, Motum omnem communis Centri Gravitatis his Actionibus sublatum esse, et Motus reliquos PQ, QR, RP, ex Centro illo jam quiescente spectatos omnino proinde se habere ac Motus priores ex eodem Centro uniformiter progrediente. Nam cum Vires novae sint ut Corpora, Velocitates quas solae generarent aequales erunt, et praeterea aequales et contrariae Velocitati qua Centrum Gravitatis movetur cum omnem hunc Motum tollant. Unde Motus reliqui post subductum Motum Centri Gravitatis iidem omnino erunt quoad hoc Centrum qui prius.

COROL. III.

Hinc et ex Corol. quinto praecedentis manifestum est, Motus Corporis unius cujusque et Centri Gravitatis reliquorum utcunque se habeant absolute, sive in eodem fiant Plano, sive non, ex communi omnium Gravitatis Centro visos, fieri semper in Plagas contrarias. Sic Motus Corporis A post subductum Motum Centri Gravitatis, fit in Plagam PQ cum tamen (per modo citatum Corol.) commune Gravitatis Centrum reliquorum B et C feratur in Plagam QP, et par est Ratio reliquorum, ut ex Contemplatione Figurae PQR P satis liquebit.

COROL.

COROL. IV.

Qua Ratione retulimus Motus partium Syſtematis ad commune Centrum Gravitatis ſubducendo Motum Centri ex Motibus partium, eâdem inveniri poſſunt Motus reſpectivi alterius cujuſcunque Centri uniformiter et in directum progredientis; nimirum quaerendo Vires $L l$, $M m$, $N n$, in plagam contrariam applicandas, quae ſemel et ſimpliciter agentes generare poſſint in Corporibus ſingulis Velocitatem Centri; et notando Motus reliquos juxta praecſcriptum Theorematis.

In Scholium ad LEGES MOTUS.

Demonſtrationem *Newtonianam*, quod Graviorum Deſcenſus ſint ut Quadrata Temporum, ſic explico.

Cadant e Quiete duo Corpora, ſintque Tempora deſcenſuum (Fig. 57.) AB et ab : Accipiantur in his Temporibus duo Momenta PQ et pq , totis proportionalia, et ſimiliter ſita, hoc eſt, ſit AP ad ap ut AB ad ab , et Velocitas in Instanti P erit ad Velocitatem in Instanti p ut Tempus AP ad Tempus ap , hoc eſt, ut Tempus AB ad Tempus ab ; ergo Velocitas in Instanti P et Momentum PQ ſimul, erit ad Velocitatem in Instanti p , et Momentum pq ſimul, ut AB^2 ad ab^2 . Sed Spatiolum a Corpore cadente, Momento PQ emenſum, eſt ad Spatiolum emenſum Momento pq ut Velocitas in P ad Velocitatem in p , atque ut Momentum PQ ad Momentum pq conjunctim:

R

Ergo

Ergo Spatium emensum Momento PQ est ad Spatium emensum Momento pq ut AB^2 ad ab^2 . Dividantur jam Tempora AB et ab in Momenta quam minima, numero quidem aequalia sed Magnitudine totis proportionalia; et cum Spatia Momentis singulis similiter fitis descripta sint ut Quadrata Temporum, erunt Spatia tota in hac Ratione *.

* Hic Demonstrationem tradit Cl. Auctor Regulae Newtonianae de Motibus ex collisione Corporum Aestimandis per Experimenta tanquam in Vacuo instituta; sed prolixam nimis, quum paucis ea res absolvi poterat.

Arcus scilicet RV quatuor Resistentias complecti censendus est, fere arithmetice proportionales, sed quarum duabus duae sunt Situ dissimiles: nam Arcus ille, quem pro mensura Resistentiae per dimidiam Oscillationem genitae usurpare licet, infra punctum unde demittitur Pendulum locandus est, et supra punctum ad quod ascenditur. Quo igitur utriusque Situs ratio babeatur, Arcum ST per quem Resistentiarum pars quarta mixtim exponitur, nec totum infra, nec totum supra medium Arcus RV punctum statuere convenit, sed in ipso Arcus medio: Sic enim, neque alias, Resistentiam utramvis quam patitur Corpus descendendo ab S ad A , vel ascendendo ab A ad T , aequo jure metietur: est enim ST utrique Arcui, SA et AT , communis; dum interea horum Arcuum semi summa quartae parti Arcuum a Pendulo descriptorum aequalis manet.

Quadratura

*Quadratura Parabolae Apolloniae, ostendens usum
Lemmatum quatuor primorum Libri primi
Principiorum, ut et usum Rationum primarum
et ultimarum in Ratiociniis Geometricis.*

Sit AMm (Fig. 58.) Parabola cujus Parameter t , Axis APp , ordinatim applicatae MP , mp :
Quaeritur Spatium parabolicum APM .

Compleantur Parallelogramma $APMQ$,
 $Apmq$, occurrantque MQ , MP ipsis mp , mq in
 R et S respective, eritque primo $t \times AP = \overline{PM}^2$,
et $t \times Ap = \overline{pm}^2$; quare $t \times Pp = \overline{mp}^2 - \overline{MP}^2 = \overline{PS}^2$
 $- \overline{PM}^2 = \overline{PS} + \overline{PM} \times \overline{PS} - \overline{PM} = \overline{PS} + \overline{PM} \times \overline{MS}$,
hoc est, Rectangulum $t \times Pp$ aequatur Rectangulo
 $\overline{PM} + \overline{PS} \times \overline{MS}$; et Solidum $AP \times t \times Pp$ Solido
 $AP \times \overline{PM} + \overline{PS} \times \overline{MS}$, vel Solidum $\overline{PM}^2 \times Pp$
Solido $\overline{PM} + \overline{PS} \times AP \times \overline{MS}$. Sed $\overline{PM} \times Pp$
aequatur Rectangulo PR , et $\overline{PM}^2 \times Pp$ aequatur
Rectangulo PR ducto in Rectam PM . Similiter
 $\overline{PM} + \overline{PS} \times AP \times \overline{MS}$ aequatur Rectangulo QS
ducto in Rectam $\overline{PM} + \overline{PS}$; quare Rectangulum
 PR est ad Rectangulum QS ut $\overline{PM} + \overline{PS}$ ad PM .
Manentibus MP , MQ , minuantur Latitudines
Rectangulorum PR , QS , *ad infinitum*, et Ratio
ultima Parallelogrammorum evanescentium erit
 $2 PM$ ad PM five 2 ad 1 : Erit ergo summa ultima
Parallelogrammorum omnium PR ad summam ulti-
mam Parallelogrammorum omnium QS ut 2 ad 1 ,
hoc est, Spatium internum APM duplum erit Spatii
externi AQM ; adeoque Spatium internum APM
subsesquialterum erit totius Spatii PQ . Q. E. I.

De Rationibus primis et ultimis.

Quaquam Magnitudines variabiles considerari possunt, vel tanquam ex nihilo nascentes, vel tanquam in nihilum evanescentes; fieri tamen potest, imo saepenumero fit, ut harum Rationes neque nascantur neque evanescant, neque ultra certam ac determinatam Magnitudinem excrescant, sed finitae existant; etiam tum cum Magnitudines, quarum illae sunt Rationes vel in nihilum abiere, vel in immensum excrevere; et hae Rationes eae ipsae sunt, quas primas nascentium, vel ultimas evanescentium sub diverso considerandi modo nuncupamus, vel etiam ultimas Rationes Magnitudinum auctarum in infinitum. Sunt itaque Rationes primae et ultimae, non Rationes Magnitudinum primarum et ultimarum, quales non dantur in rerum Natura, non Rationes Magnitudinum finitarum, aut vere existentium; sed Rationum illarum Limites, ad quos nimirum Rationes Magnitudinum sine Limite crescentium vel decrescantium propius accedere possunt quam pro data quavis differentia; nunquam autem transgredi, neque attingere quidem, nisi tunc cum Magnitudines, quarum sunt Rationes vel immensae evaserint, vel nullae. Haec omnia utcumque conceptu difficilia prima fronte videantur, unico tamen exemplo inter alia quae passim plurima occurrunt, satis illustrari posse existimo.

Exponantur

Exponantur itaque duae Quantitates A et B, quarum sit $A=4xx+3x$ et $B=2xx+x$, variables quidem propter variabilem x , et immensae vel nullae prout Radix earum x infinita fuerit, vel nulla; dico jam ultimam Rationem evanescentis A ad evanescentem B, eam ipsam fore quam habet 3 ad 1; ultimam autem Rationem infinitae A ad infinitam B, eam fore quam habet 2 ad 1. Etenim cum sit $A=4xx+3x$, et $B=2xx+x$, erit A ad B (dividendo per x) ut $4x+3$ ad $2x+1$: Est autem haec Ratio major quam $4x+2$ ad $2x+1$ vel 2 ad 1, et minor quam Ratio $6x+3$ ad $2x+1$ vel 3 ad 1. Evanescat x et simul evanescentibus $4x$ et $2x$ erit A ad B ultimo ut 3 ad 1. Rursus cum sit A ad B universaliter, ut $4x+3$ ad $2x+1$, vel (dividendo per x) ut $4+\frac{3}{x}$ ad $2+\frac{1}{x}$; augeatur x in infinitum, et evanescentibus $\frac{3}{x}$ et $\frac{1}{x}$, erit A ad B ultimo ut 4 ad 2, vel 2 ad 1; nunquam ergo erit A ad B, vel in Ratione 3 ad 1, vel in Ratione 2 ad 1, sed in Ratione intermedia; et hae Rationes, nempe 3 ad 1 et 2 ad 1, sunt Limites certi, et jam determinati, Rationum omnium quas habere potest Quantitas A ad Quantitatem B; inter quos quidem Limites, Rationes illae indeterminatae tanquam inter cancellos cohibentur, et ad quos propius accedere possunt quam pro data quavis differentia. (Vide *Newtonum* sub finem Sectionis primae.)

De Curvatura et Angulo Contactus.

DEFINITIO.

Si ad Curvae alicujus particulam seu Arcum quam minimum, et ad easdem partes, applicetur Arcus quam minimus Circularis, qui cum priore, quantum fieri potest, congruat; horum Curvatura eadem esse dicitur: Sin alter alterum in sinu suo complectatur, exterioris Curvatura minor esse dicitur, interioris major: Et universaliter, *Quantitas Curvaturae in loco quovis ex reciproca Ratione diametri Circuli aequae curvi semper aestimatur*; unde Curvatura finita esse dicitur, ubi Diameter Curvaturae, hoc est, Diameter Circuli aequae curvi est finita.

THEOREMA I. Fig. 59.

Si per Curvae alicujus tria Puncta A, B, C, ad invicem continuo accedentia, et in loco B ultimo coeuntia, transeat semper Circulus D; dico Circulum illum D eandem ultimo Curvaturam habiturum quam Curva in loco B.

DEMONSTRATIO.

Habeat enim (si negas) alius Circulus E eandem Curvaturam, congruatque quantum fieri potest, cum Curva in loco B, et Circulus iste E vel transibit per tria illa Puncta A, B, C, vel non; transeat ergo, et duo Circuli D et E se mutuo secabunt in pluribus quam duobus Punctis, contra quod demonstravit *Euclides* in decima Propositione Libri tertii Elementorum. Si non transeat, ergo Circulus

culus D magis congruet cum Curva in loco B quam E Circulus ejusdem Curvaturae (hoc est) quam Circulus maxime congruus Q. E. A. Habet itaque Circulus D Curvaturam praefinitam. Q. E. D.

COROLLARIUM.

Si ex tribus Punctis A, B, C, duo tantum A et B coierint, immoto manente tertio C, Circulus quidem D tanget Curvam in loco B, sed eandem Curvaturam non habebit necessario, nisi et tertium C cum reliquis duobus A et B coaluerit.

SCHOLIUM.

Quemadmodum dantur Circuli numero infiniti, qui per duo data Puncta transire possunt, unicus autem qui per tria; ita dantur Circuli numero infiniti qui eandem Curvam in eodem loco contingere possunt, unicus autem qui eandem habebit Curvaturam, congruetque cum Curva in loco Contactus. Porro ex jam dictis et demonstratis, facile constabit, particulasquam minimas Curvarum omnium ubi Curvatura est finita, iisdem omnino affectionibus gaudere quoad ipsarum *Chordas, Tangentes, Sagittas, &c.* quibus Arcus quam minimi Circulares ejusdem Curvaturae; harum itaque nonnullas ex natura Circuli, tanquam ex proprio fonte deductas, praesertim quas in *Philosophia Newtoniana* maximè usui fore praevidero in medium proferre et explicare non pigebit. Sunt autem hujusmodi.

I.

Si Circulum ABC (Fig. 60.) tetigerit Recta quaeprim AD , et per hujus Circumferentiam moveatur Punctum B versus Punctum immotum A , coeatque ultimo cum eodem, et si per immotum Punctum A , et motum B , transire semper intelligatur Chorda AB ; dico Angulum BAD inter Chordam AB , et Tangentem AD , ultimo minorem fore dato quovis Rectilineo. Nam Angulus iste BAD semper aequalis erit Angulo ACB in alterno Segmento, qui coeuntibus A et B evanescit.

II.

Si a Puncto B agatur utcumque subtenfa BD , efficiens Angulum quemlibet finitum BDA cum Tangente AD , dico Chordam AB , tangentem AD , et Arcum AB fore ultimo in Ratione aequalitatis. Posita enim AC subtenfae BD parallela, similia erunt Triangula CAB , ABD ; unde AB et AD et $AD \div DB$ erunt ad invicem ut CA et CB et $CB \div BA$. Coeat jam Punctum B cum Puncto A , eruntque CA et CB et $CB \div BA$ ultimo in Ratione aequalitatis; quare AB et AD et $AD \div DB$ erunt etiam ultimo in Ratione aequalitatis; et à fortiori AB et AD et Arcus AB erunt ultimo in Ratione aequalitatis.

III.

Quapropter, in omni de Rationibus ultimis computatione, hae lineae pro se invicem usurpari possunt.

I

IV.

IV.

Si ducatur Chorda BFG , Tangenti AD parallela, secans Chordam AC in F , Sagitta AF ultimo bifecabit et Arcum BAG et Chordam BFG ; nam ob parallelas AD , BG , est Punctum A in medio Arcus BAG , et si AF et BD etiam parallelae ponantur, erit BF vel AD aequalis Arcui AB ; et pari Ratione erit FG aequalis Arcui AG .

V.

Subtensa BD , et huic aequalis et parallela Sagitta AF , erunt ut Quadratum Arcus BA , vel hujus dupli BAG directe, atque ut Chorda AC inverſe; ſi modo Chorda illa AC ſit parallela ſubtenſae BD , et Chorda BG , Tangenti AD . Nam ob dicta Triangula ſimilia CAB , ABD , eſt CA ad AB ut AB ad BD : Ergo univerſaliter eſt BD vel AF aequalis Quadrato Chordae AB applicato ad Chordam AC ; et evaneſcens Chorda aequalis eſt Arcui evaneſcenti,

VI.

Adeoſque in Figura quavis curvilinea, ſi detur et Angulus ad D et Curvatura ad A , ſubtenſa evaneſcens BD et Sagitta AF erunt ut Quadratum Arcus BA vel Arcus BAG .

VII.

Quod ſi plures fuerint Arcus quam minimi, et viciniſſimi, erunt horum Quadrata ut Sagittae illae
quae

quae Chordas bifecant, et ad datum Punctum convergunt. Nam ob parvitatem et vicinitatem Arcuum, erit eadem omnium Curvatura, Tangentes quasi in eadem sunt Recta, et omnium Sagittae quasi parallelae.

VIII.

Si ex Curvae alicujus Punctis A et B, ducantur utcunque Rectae AF, BF, ita tamen ut BF parallela fit Tangenti in A, et si producat A F ad C, ita ut Longitudo AC aequalis sit ei quae ultimo fit, applicando Quadratum vel Arcus evanescens AB, vel Chordae AB, vel Ordinatae BF ad interceptam AF, Circulus qui tangit Curvam in A transitque per C, eandem habebit Curvaturam cum Curva in loco Contactus.

IX.

Vel si tangat Recta AD Curvam aliquam in loco A quam secat subtensa BD in B, et si ex Punctis A et B ducantur Rectae AC, BC efficientes Angulos ABC, BAC Angulis ADB, ABD aequales respective, Rectae illae AC, BC ultimo concurrent in peripheria Circuli aequae Curvi, et cum altera illa Curva congruentis in A.

X.

Postremo, si in Figura quavis curvilinea, Curvatura ad A non detur, si tamen Recta AC subtensae BD parallela terminetur ad peripheriam Circuli aequae Curvi, hoc est, si Arcus quam minimus AB compleatur in Circulum aequae Curvum

Curvum ABC , subtensa evanescens BD aequalis erit Quadrato Arcus evanescentis AEB applicato ad Rectam illam AC .

Theorema sequens respicit Arcus quam minimos Curvarum, quorum Curvaturae infinite majores sunt, vel infinite minores circularibus. Vide Scholium ad finem Sectionis primae Principiorum.

THEOREMA II. Fig. 61.

Si Curvae AB , AC , et Recta AD se mutuo contingant in loco A , actâque secundum quamcunque legem Recta DBC , fuerit DB , vel universaliter, vel saltem evanescens, ut $\overline{AD}^m$, et CD ut $\overline{AD}^n$, existente m majore quam n : Dico Angulum Contactus BAD infinite minorem fore Angulo CAD , et idcirco nullam Lineam Curvam ejusdem generis cum AC duci posse inter Curvam AB et Tangentem suam AD .

DEMONSTRATIO.

Esto enim $BD = \frac{\overline{AD}^m}{p}$, et $CD = \frac{\overline{AD}^n}{q}$, positis p et q finitis et constantibus, erit BD ad CD ut $\frac{\overline{AD}^m}{p}$ ad $\frac{\overline{AD}^n}{q}$ vel ut $\overline{AD}^m$ ad $\overline{AD}^n \times \frac{p}{q}$, vel ut $\frac{p}{\overline{AD}^{m-n}}$ ad $\frac{q}{1}$. Evanescat jam AD et simul $\overline{AD}^{m-n}$, et BD jam erit ad CD ut Quantitas evanescens $\overline{AD}^{m-n}$ ad Quantitatem finitam $\frac{p}{q}$, vel ut finitum ad infinitum: Est itaque subtensa evanescens BD infinite minor subtensa evanescente CD , et propterea Angulus BAD Angulo CAD , Q.E.D.
Demon-

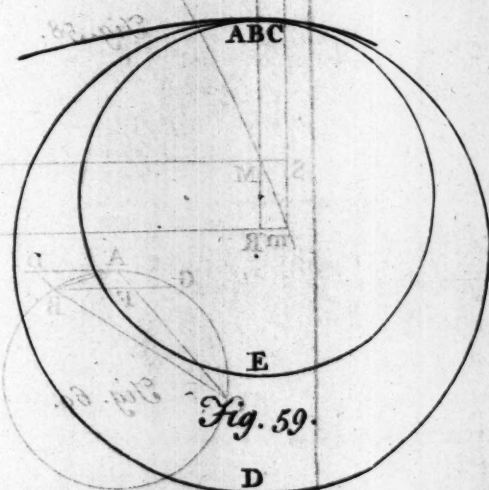
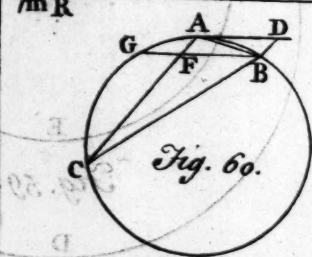
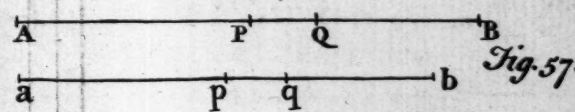
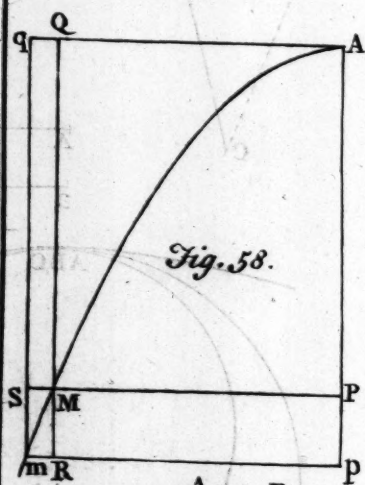
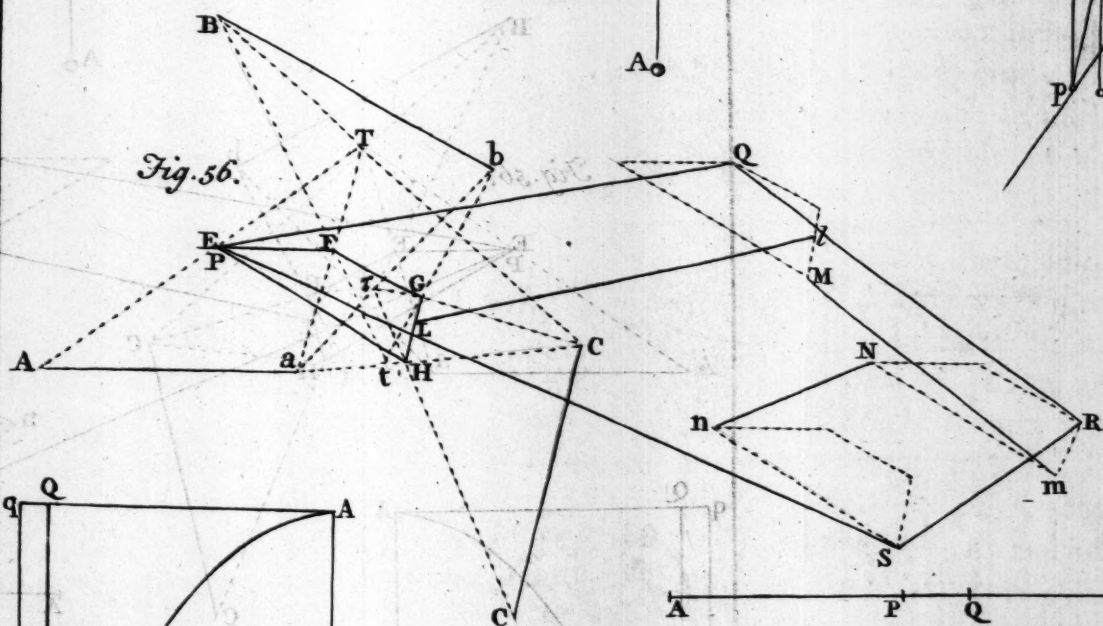
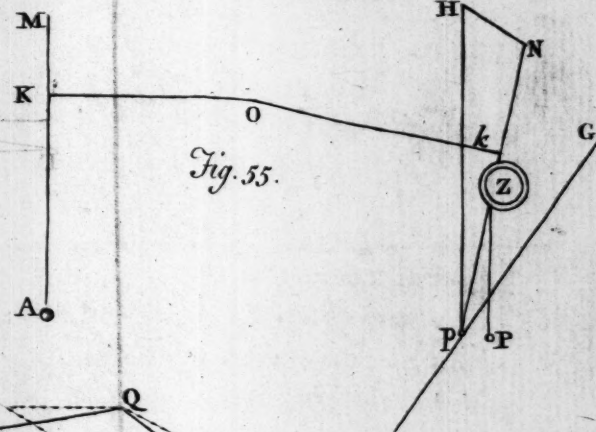
*Demonstratio Corollarii octavi Propositionis
quartae.*

*Si duo Corpora in Orbibus similibus revolvantur,
Viribus centripetis ad Centra similiter posita ten-
dentibus : dico Vires hasce centripetas in locis simi-
libus esse ut Arcuum quam minimorum et simul de-
scriptorum Quadrata applicata ad Radios ab his
locis ad Centra Virium ductos. Fig. 62.*

Sint enim S et s Centra Virium, B et b Orbi-
tarum loca similia, ABC et abc Arcus quam
minimi simul descripti, quorum media Puncta
sunt B et b ; et compleantur hi Arcus in Circulos
aeque curvos $ABCE$ et $abce$, secantes Radios
 BS et bs (si opus productos) in E et e ; et cum
Arcus ABC et abc simul describantur, erit Vis
centripeta in B ad Vim centripetam in b , ut Arcus
 ABC Sagitta, quae Chordam bisecat, et per
Centrum Virium transit, ad similem Sagittam
Arcus abc : Sed hae Sagittae sunt ad invicem ut
 $\frac{ABC^2}{BE}$ ad $\frac{abc^2}{be}$, per ea quae supra de Curvatura
scripsimus; ergo Vis centripeta in B est ad Vim
centripetam in b , ut $\frac{ABC^2}{BE}$ ad $\frac{abc^2}{be}$. Sed in Fi-
guris similibus, lineamenta omnia similia sunt
proportionalia; adeoque BE est ad be ut BS ad bs :
Substituuntur ergo BS et bs pro BE et be , et Vis
centripeta in B erit ad Vim centripetam in b ut
 $\frac{ABC^2}{BS}$ ad $\frac{abc^2}{bs}$. Q. E. D.

COROL.

Plate X.



COROL. I.

Vires centripetae in locis similibus sunt ut Quadrata Velocitatum in his locis directe, atque ut Radii ad Centra Virium ducti inverse. Nam Velocitates sunt ut Arcus simul descripti.

COROL. II.

Sunt etiam Vires centripetae similibus in locis ut Radii ad Centra Virium ducti directe, atque ut Quadrata temporum periodicorum inverse. Referant enim ABC et abc jam non amplius Arcus quam minimos simul descriptos, sed similes similibus Figurarum particulas, temporibus quam minimis T et t descriptas; et Velocitas in B erit ad Velocitatem in b ut $\frac{ABC}{T}$ ad $\frac{abc}{t}$, nempe ut Spatia descripta directe, et tempora inverse. Sed Arcus ABC est ad Arcum sibi similem abc ut Longitudo BS ad similem Longitudinem bs ; quare Velocitas in B est ad Velocitatem in b ut $\frac{BS}{T}$ ad $\frac{bs}{t}$, et Quadratum Velocitatis in B ad Quadratum Velocitatis in b ut $\frac{BS^2}{T^2}$ ad $\frac{bs^2}{t^2}$. Sed per ultimum Corollarium, Vires centripetae in B et b sunt ut Quadrata Velocitatum in his locis directe, atque ut Radii ab iisdem locis ad Centra Virium ducti inverse: Ergo Vis centripeta in B est ad Vim centripetam in b ut $\frac{BS}{T^2}$ ad $\frac{bs}{t^2}$. Verum enim vero, cum similes similibus Arearum partes $SABC$,

SABC, *sabc* sint ad Areas totas in eadem Ratione, erunt et tempora *T* et *t* in eadem Ratione ad tempora tota periodica: Quare Vires centripetae in locis similibus *B* et *b* erunt ut Radii ab iisdem locis ad Centra Virium ducti directe, atque ut Quadrata Temporum periodicorum inverse.

Reliqua consequuntur, ut ex iisdem demonstratis in Circulo.

Demonstratio Corollarii noni Propositionis quartae.

Fig. 63.

Si Corpus revolvatur uniformiter in Orbe circulari, Vi centripeta ad Centrum tendente percurrentes Arcum quemlibet AB tempore t: Dico Arcum illum AB medium fore Proportionalem inter AE Diametrum Circuli et s descensum rectilineum eodem tempore t confectum, si Corpus revolvens Motu omni circulari privatum recta versus Centrum dimitteretur; eadem urgente Vi centripeta qua prius in Orbe circulari coercebatur.

Nam data Vi centripeta, Descensus *s* erit ut Quadratum temporis *t*, hoc est, ob suppositum uniformem Corporis Motum, ut Quadratum Arcus *AB*, vel ut $\frac{AB^2}{AE}$; unde si in casu aliquo particulari, constiterit Longitudines *s* et $\frac{AB^2}{AE}$ aequales esse, semper aequales erunt. Tangat Recta *AD* Circulum in *A*, agaturque Subtensa *BD*

BD Diametro AE parallela, et si Arcus AB
ponatur infinite parvus, erit Subtensa $BD = \frac{AB^2}{AE}$,

ut ex iis constat quae de Curvatura demonstravi-
mus. Sed in hoc casu, est etiam BD aequalis
Spatio s: nam tempore quam minimo; conside-
rari potest Vis centripeta tanquam agens secun-
dum Rectas Diametro AE parallelas: Quare in
illo casu, ubi Arcus AB infinite parvus est, Lon-
gitudines s et $\frac{AB^2}{AE}$ aequales sunt; ergo aequales
erunt universaliter. Q. E. D.

C O R O L L A R I U M.

Et hinc rursus patet, Vires centripetas in Or-
bitis circularibus, quae ad horum Orbium Centra
tendunt, esse ut Quadrata Arcuum simul descrip-
torum directe, atque ut Circulorum Diametri vel
Semidiametri inverse. Nam hae Vires erunt ut
Descensus rectilinei quos eodem tempore efficiunt.

*Ad illustrandum Corollarium nonum Propositionis
quartae.*

P R O B L E M A.

*Referat s Spatium à Gravi è Quiete demisso, tempore
unius minuti secundi descriptum, d Diametrum
Terrae, et seponatur omnis ex Aere Resistentia;
quaeritur quanta cum Velocitate Projectile aliquod
ex editiore in Terrae Superficie loco, et secundum
Rectam Horizonti parallelam emittendum sit ut fiat
Planeta*

Planeta secundarius, hoc est, ut instar Lunae nostrae, Circulum Terrae concentricum uniformi cum Motu percurrat.

Pone factum, et Arcus singulis Minutis secundis descriptus, medius erit Proportionalis inter Diametrum d et Spatium s . Innotescunt autem et d et s ex notis Terrae Magnitudine, et Longitudine Penduli ad Minuta secunda oscillantis. Emittatur itaque Projectile tanta cum Velocitate, quanta sufficiat ad Longitudinem $\sqrt{ds}$ Spatio unius Minuti secundi uniformiter percurrentem, et solvetur Problema.

COROLLARIUM.

Si desideretur tempus periodicum, dicendum est, ut $\sqrt{ds}$ ad totum Terrae Ambitum ita tempus unius Minuti secundi ad tempus quaesitum.

SCHOLIUM.

Ambitus Telluris est 123249600 pedum Parisiense; unde Diameter ejus d est eorundum pedum fere 39231600; sed et Spatium s ad eandem Mensuram exactum, est $15\frac{1}{12}$ pedum; ergo prodit tandem $\sqrt{ds}$ pedum Gallicorum 24326, Anglicorum 25982, hoc est, milliarius Anglicorum 4.92083. Tanta itaque esse debet Velocitas projectilis, quanta sufficiat ad Spatium s fere milliarius tempore Minuti unius secundi uniformiter percurrentem; unde tempus periodicum projectilis erit unius horae, 24 minutorum primorum, 27 secundorum. Ex his autem manifestum est, quod

quod si Terra nostra Revolutiones singulas circa proprium Axem Spatio unius horae, 24 minutorum primorum, 27 secundorum perageret, Partes omnes juxta Aequatorem sitae, Gravitationem suam omnino amitterent. Quod si Terra incitatioe adhuc provolveretur Motu, Aer omnis et Aqua et Partes omnes juxta Aequatorem, Terrae non adhaerentes, in Spatia circumjacentia penitus avolarent, et ibi orbitas ellipticas umbilicos alteros in centro telluris habentes, motu suo describerent, ut in sequentibus luculentius constabit.

THEOREMA.

Si Corpus in Circuli peripheria, cujus Diameter est d, uniformiter Motum, ope Fili teneatur, ne ulterius excurrat, absolvatque Arcum quemlibet p, quo tempore idem vel aliud grave è Quiete demissum descenderet per Spatium s: dico Vim centrifugam Corporis revolventis fore ad Vim Gravitationis ut $p p$ Quadratum Arcus descripti ad Rectangulum $d s$, seu (quod eodem recidit) dico Tensionem Fili ex Vi centrifuga Corporis oriundam fore ad Tensionem ejusdem ex eodem pondere libere suspensio in hac Ratione.

Nam Vis centrifuga Corporis revolventis aequalis est Vi centripetae qua versus Centrum urgetur, quaque uniformiter urgente, si nihil impediret, Corpus descenderet per Spatium $\frac{p^2}{d}$, quo tempore descendit grave per Spatium s ex Hypothesi: Ergo Vis centripeta vel centrifuga Coporis revolventis est ad Vim Gravitationis ut $\frac{p^2}{d}$ ad s vel ut p^2 ad $d s$.
 Q. E. D. S In

In ejusdem Propositionis SCHOLIUM.

Si Polygonum detur Magnitudine, et Velocitas non detur, Vis ictum singulorum erit ut Velocitas, hoc est, ut Arcus rectilineus dato Tempore descriptus; et numerus ictuum dato Tempore erit etiam ut Arcus rectilineus: Quare Vis et numerus ictuum simul, hoc est, summa Virium sive Vis centrifuga, erit ut Quadratum Arcus rectilinei dato Tempore descripti. Haec ita sunt ex Hypothesi, quod Polygonum detur Magnitudine, et Velocitas non detur: Detur jam Velocitas, et etiam Polygonum specie, non autem Magnitudine; et Vis ictuum dabitur: Sed numerus ictuum jam erit reciproce ut Semidiameter Circuli cui inscribitur Polygonum: Nam quo major est Semidiameter Circuli, eo minor est numerus Angulorum in Arcu rectilineo datae Longitudinis; ergo quo major est Diameter Circuli, eo minor erit numerus Reflectionum dato Tempore confectarum; quare si detur Velocitas, numerus Reflectionum erit reciproce ut Semidiameter Circuli: Quare si neque Polygonum detur Magnitudine, neque Velocitas, summa Virium in dato Tempore erit ut Quadratum Arcus rectilinei dato Tempore descripti applicatum ad Circuli circumscripti Semidiametrum.

PROBLEMA.

Efficere ut Corpus aliquod in Peripheria dati Circuli uniformiter moveatur, et Vim determinare quae sufficiat ad istiusmodi Motum conservandum.

SOLUTIO. Fig. 64.

Moveatur Corpus primo in perimetro Polygoni cujuscunque aequilateri $ABCDEFGHI$ dato Circulo inscripti, cujus Centrum sit S , et producatur Latus quodvis AB ad c , ita ut aequales sint AB, Bc ; et Corpus ubi ad B pervenerit, (si nihil impediret) Longitudinem Bc eodem Tempore describeret quo Latus AB antea descripsit: Verum quo minus hoc fiat, agat Vis quaedam impulsu unico ea Quantitate ac Determinatione ut Corpus hac Vi impulsu de Recta Bc in Rectam BC detorqueatur, eamque praeterea eodem Tempore percurrat, quo antea Latus AB percurrerat: Et si Corporis in Latere AB moti, exponatur Vis insita per Longitudinem illam AB vel Bc , et simul compleatur Parallelogrammum $BcCK$, Longitudo BK vel cC , novi hujus impulsus tum Quantitatem tum Determinationem exhibebit: Et si ad Angulos singulos Corpus hoc modo de Latere in Latus deflectatur, perget idem in Polygoni hujus perimetro uniformi cum Velocitate moveri. Connectantur jam SA, SB , et cum Anguli SAB, ASB simul aequantur Angulo SBc ; (per prop. xvi. l. 1. El.) si auferantur aequales Anguli SAB, SBC , manebit Angulus CBc aequalis Angulo

$S 2$ $ASB;$

ASB; unde cum aequales ponantur BC, Bc, similia erunt Triangula isoscelia ASB, CBc; unde Recta Cc parallela erit ipsi SB, eritque ad CB vel AB ut AB ad AS: Tendet itaque Vis BK ad Centrum S, et aequalis erit Longitudini

$$\frac{AB^2}{AS}$$

Supponamus iisdem impulsibus accelerantibus, quibus hoc Corpus in perimetro Polygoni retinetur, et similiter agentibus, urgeri Corpus aliud in directum, Motu aequabiliter accelerato; et constat dum Corpus primum percurrit Longitudinem AB, Corpus illud alterum primo suo impulsu

conficere Longitudinem $\frac{AB^2}{AS}$, vel $\frac{2AB^2}{2AS}$, vel $\frac{AB \times ABC}{2AS}$. Similiter dum Corpus primum

percurrit Longitudinem ABC, Corpus alterum duobus jam impulsibus successive agentibus feretur

per Longitudinem $\frac{AB^2 + 2AB^2}{AS}$ vel $\frac{6AB^2}{2AS}$ vel $\frac{ABC \times ABCD}{2AS}$; et porro dum Corpus primum

Longitudinem ABCD conficit, Corpus alterum conficiet Longitudinem $\frac{AB^2 + 2AB^2 + 3AB^2}{AS}$,

vel $\frac{12AB^2}{2AS}$, vel $\frac{ABCD \times ABCDE}{2AS}$. Et uni-

versaliter dum Corpus primum Arcum quemvis rectilineum absolvit, Corpus alterum impelletur per Longitudinem quae sit ad hunc Arcum ut
Arcus

Arcus iste unico Polygoni Latere auctus, ad Circuli Diametrum. Augeatur jam numerus et minuatur Longitudo Laterum in infinitum, ita nempe, ut Polygonum coincidat omni ex parte cum Circulo circumscripto, et Vis acceleratrix jam continue efficiet ut Longitudo a Corpore aequabiliter accelerato confecta, sit ad Arcum circularem a Corpore uniformiter gyrante interea descriptum, ultimo ut Arcus iste ad Circuli Diametrum. Q. E. I.

COROL. I.

Hinc e converso, si detur Vis centripeta Corporis in dato Circulo revolventis, facile invenietur Motus Corporis in hoc Circulo, Nam Arcus dato quovis Tempore descriptus, medius erit proportionalis inter Circuli Diametrum, et Longitudinem quae Motu ab hac Vi aequabiliter accelerato pari Tempore conficitur.

COROL. II.

Spatia Motibus aequabiliter acceleratis, et e Quiete inchoatis, descripta sunt ut Quadrata Temporum ab initio Motuum, et Vires acceleratrices conjunctim. Constat enim ex Demonstratione Propositionis, Longitudines a Corpore aequabiliter accelerato confectas, fuisse ut Quadrata Arcuum a Corpore revolvente interea descriptorum, et proinde etiam ut Quadrata Temporum iisdem Arcubus proportionalium. Quod si e contra dentur Tempora, Spatia percurfa erunt ut Velocitates, adeoque ut Vires generantes.

COROL. III.

Sunt etiam Spatia illa ut Vires simul et Quadrata Velocitatum quae dictis Temporibus generantur. Nam Velocitates, caeteris paribus sunt ut Tempora.

COROL. IV.

Spatia Motibus aequabiliter retardatis, ac tandem evanescentibus descripta, sunt ut Quadrata Velocitatum sub initio Motuum.

COROL. V.

Si Tempora in partes aequales dividantur, Spatia Motibus aequabiliter acceleratis iisdem Temporibus descripta, erunt ut numeri impares 1, 3, 5, 7, 9, &c.

COROL. VI.

Velocitas Corporis Vi quavis centripeta, in Circuli peripheria revoluti, est ad Velocitatem aequali-Vi uniformiter agente, dato quovis Tempore genitam, ut Circuli Semidiameter ad Arcum eodem Tempore descriptum. Nam Velocitas Corporis in Orbe rectilineo erat ad Velocitatem alterius in directum post primum impulsu ut BC ad Cc, vel ut SB ad AB; et haec Velocitas erat ad Velocitatem postea, ut unitas ad numerum ictuum accelerantium, hoc est, ut AB ad Arcum rectilineum toto Accelerationis Tempore emensum. Quare ex aequo Velocitas prima est ad Velocitatem ultimam ut Semidiameter Circuli circumscripti ad Arcum praefinitum: et par est Ratio in Orbe circulari.

COROL.

COROL. VII.

Spatium Motu aequabiliter accelerato emensum, dimidium tantum est ejus quod uniformiter, Velocitate ultimo acquisita eodem Tempore percurri potest. Nam Spatium Motu aequabiliter accelerato emensum est ad Arcum circulare, eadem Vi centripeta, eodemque Tempore descriptum, ut Arcus iste ad Circuli Diametrum, (per Corol. I.) vel ut Semi-arcus ad Semidiametrum, Verum Spatium, Velocitate ultimo acquisita percursum erit ad eundem Arcum interea descriptum, ut Velocitas ad Velocitatem, hoc est, (per ultimum Corol.) ut Arcus totus ad Circuli Semidiametrum.

COROL. VIII.

Vires centripetae iisque aequales etiam Vires centrifugae Corporum in Circulis gyantium sunt ut Arcuum simul descriptorum Quadrata ad Circulorum Diametros vel Semidiametros applicata.

COROL. IX.

Sunt etiam hae Vires ut Quadrata Velocitatum itidem ad Circulorum Diametros vel Semidiametros applicata: Nam Velocitates sunt ut Arcus simul descripti.

COROL. X.

Postremo sunt hae Vires ut Circulorum Semidiametri directe, et Quadrata Temporum periodicorum inverse. Nam si sublato omni Motu circulari Corpora in directum iisdem Viribus cen-

tripetis uniformiter agentibus impellerentur, Vires per Corol. 2. essent ut Spatia descripta directe, et Quadrata Temporum inverse. Verum si Tempora ea sumantur quibus similes Arcus peraguntur, Spatia quae sunt ut Arcuum illorum Quadrata ad Circulorum Semidiametros applicata, jam erunt ut eadem Semidiametri. Sunt itaque vires centripetae vel centrifugae, ut Semidiametri illi directe, et Quadrata Temporum periodicorum inverse, seu (quod perinde est) sunt hae Vires reciproce ut Quadrata Temporum periodicorum ad Circulorum Semidiametros applicata.

COROL. XI.

Hinc si Tempora periodica sint ut dignitas aliqua Semidiametrorum vel Distantiarum a Centro, cujus Index est n , Vires centripetae erunt reciproce ut dignitas $2n-1$, vel directe ut dignitas $1-2n$ earundem Distantiarum. Nam Quadrata Temporum periodicorum erunt ut harum Distantiarum dignitas $2n$, adeoque horum Quadrata ad hasce Distantias applicata, quibus nempe recipiuntur Vires centripetae, erunt ut Distantiarum dignitas $2n-1$.

COROL. XII.

Velocitates etiam erunt reciproce ut Distantiarum dignitas $n-1$. Nam Velocitatis illae (per Corol. 9.) sunt in subduplicata Ratione Distantiarum et Virium centripetarum, conjunctim, hoc est, (per Corol. ult.) ut Distantiarum dignitas $1-n$ directe vel $n-1$ reciproce. Q. E. D.

THEOREMA.

THEOREMA.

Motus circularis uniformis erit, siue ab Actione Vis alicujus ad Centrum tendentis, siue a Reactione Fili alicujus non elastici, vel Superficieii Cylindricae politissimae, Corpus motum a cursu rectilineo perpetuo defleuentis, oriatur.

Patet pars prior ex Aequalitate Arearum Temporibus aequalibus circa Centrum descriptarum: Et pars posterior sic patebit.

Sunto AB, BC, (Fig. 65.) Latera contigua Polygoni cujusvis aequilateri, et Circulo inscripti cujus Centrum sit D, et jungantur AD, BD, CD, ut et AC secans Semidiametrum BD in E. Producat CBF cui perpendicularis sit AF; et ob similia Triangula ABF, ADE, erit AB ad BF ut AD ad DE. His praemissis, feratur Corpus aliquod uniformiter in Latere AB, cum Vi vel Momento AB, et impinget hoc sese in Latus BC cum Vi AF, et post impactum retinebit Momentum BF: Unde Motus Corporis in AB erit ad Motum in BC ut AB ad BF, vel ut AD ad DE; quare Motus in AB erit ad Motum amissum ad Angulum B ut AD ad BE, vel ut AD ad $\frac{AB^2}{2AD}$, vel ut $2AD^2$ ad AB^2 . Vocetur

perimeter Polygoni p ; et si Motus amissi Quantitatem ad Angulos singulos eandem esse ponamus, Motus sub initio erit ad Motus Decrementum post primam Revolutionem, ut $2AD^2$ ad $p \times AB$.
Atqui

Atqui Motus decrementum ad Angulos singulos non est idem, sed perpetuo diminuitur, et in eadem Ratione qua Motus ipse diminuitur: Nam decrementum Motus ad Angulum quemvis est ad Motum decrefcentem, in data Ratione BE ad BD ; quare decrementum Motus post unicam Revolutionem factam, minus est eo quod exponitur per Quantitatem $p \times AB$. Augeatur jam numerus, et minuatur Longitudo Laterum AB , BC , et Quantitas $p \times AB$ evanescet, et Motus jam fiet uniformis; et par est Ratio in omnibus Curvaturis quae non sunt infinite majores circularibus.

In SCHOLIUM Prop. 8. Fig. 66.

Revolvatur Corpus aliquod P in perimetro Ellipseos APB Vi centripeta tendente ad Punctum R, adeo longinquum ut Rectae omnes RP pro parallelis haberi possint: Quaeritur lex Vis centripetae tendentis ad Punctum R.

Ducatur Semidiameter CA Rectis RP parallela, cui productae occurrat in G Recta PG tangens Orbitam in loco P , et si Vis centripeta ad Centrum C tenderet, foret haec Vis ut CP Distantia Corporis a Centro, (per hujus Prop. 10.) et proinde per Solidum $\overline{RP^2} \times CP$ recte exponitur, cum detur $\overline{RP^2}$: Verum si Vis centripeta ad C tendens exponatur per $\overline{RP^2} \times CP$, Vis centripeta ad Punctum R tendens exponetur per $\overline{CG^3}$ (per Corol. 3. praecedentis.) Ergo Vis centripeta tendens ad R

est ut $\overline{CG}^3$. Ducatur Semidiameter CB Semidiametro CA conjugata, ut et ordinatim applicatae PQ, PM; PQ ipsi CA, et PM ipsi CB; et ex natura Tangentis erit CG ad CA ut CA ad CQ vel PM; quare CG erit reciproce ut PM; ergo Vis centripeta in loco P erit reciproce ut Cubus Ordinatae PM. Q. E. I.

Abeat jam Figura Elliptica in Parabolam, et Ordinata PM jam pro invariabili haberi potest: Evadet enim PM in hoc casu infinita; et Variationes finitae Quantitatum infinitarum non plus valent quam Variationes infinite parvae Magnitudinum finitarum: Ergo in Parabola Vis centripeta secundum Rectas parallelas agens, est invariabilis; si Vis centripeta secundum Rectas parallelas agens, sit invariabilis, Figura descripta erit Parabola, quod est Theorema Galilaei.

Applicari potest haec Demonstratio etiam ad Hyperbolam, nominando et construendo duntaxat Hyperbolam pro Ellipfi.

Ad Prop. 9. LEMMA. Fig. 67.

Constituatur Series Triangulorum confimilium SAB, SBC, SCD, SDE, SEF, &c. aequales Angulos habentium ad Punctum S, et Figurae sic constitutae Proprietates erunt quae sequuntur.

1. Anguli omnes SAB, SBC, SCD, SDE, SEF erunt aequales.
2. Sectores omnes rectilinei sub aequalibus Angulis ad Punctum S erunt Figurae similes.

Sic

Sic Sector rectilineus $SABC$ similis erit Sectori rectilineo $SDEF$, cum ex partibus similibus constituentur.

3. Lineae omnes in Sectori $SABC$ erunt ad Lineas omnes similiter fitas in Sectori $SDEF$ in data Ratione SA ad SD .
4. Augeatur jam numerus et minuatur Longitudo Rectarum AB, BC, CD, DE, EF ; et Figura rectilinea $SABCDEF$ migrabit in Spiralem aequiangulum secantem Radios omnes $SA, SB, \&c.$ in Angulis aequalibus; et Sectores omnes jam mixtilinei sub aequalibus Angulis ad Punctum S contenti erunt adhuc Figurae similes.

Haec est natura *Spiralis aequiangulae*.

Ad Corol. 2. Prop. 10. Fig. 68.

Sunto $APa, A pa$ Ellipses, communem habentes Axem majorem Aa ; et si ducatur Ordinata communis QpP , erit QP ad Qp in data Ratione; nimirum in Ratione Axis minoris Figurae APa ad Axem minorem Figurae $A pa$.

Sit enim CB Axis minor Figurae APa , et Cb Axis minor Figurae $A pa$, et erit $\overline{PQ}^2$ ad Rectangulum AQa ut $\overline{CB}^2$ ad $\overline{CA}^2$, ex conicis; ergo vicissim $\overline{PQ}^2$ erit ad $\overline{CB}^2$ ut Rectangulum AQa ad $\overline{CA}^2$; et pari Ratione $\overline{pQ}^2$ erit ad $\overline{Cb}^2$ ut AQa ad $\overline{CA}^2$; ergo $\overline{PQ}^2$ erit ad $\overline{CB}^2$ ut $\overline{pQ}^2$ ad $\overline{Cb}^2$; et vicissim $\overline{PQ}^2$ ad $\overline{pQ}^2$ ut $\overline{CB}^2$ ad $\overline{Cb}^2$, et PQ erit ad pQ in data Ratione CB ad Cb . Q. E. D.

Revolvantur

Revolvantur jam in his Ellipsibus Corpora duo P et p , Vi centripeta tendente ad Centrum commune : Dico Velocitatem Corporis P in Vertice principali A fore ad Velocitatem Corporis p in eodem loco in data Ratione QP ad Qp . Moveatur enim Punctum Q una cum Ordinata communi QpP usque ad Verticem A ; et si Corpora P et p simul exeant de loco A , simul pervenient ad Ordinatam communem QpP ; id adeo ob Vires acceleratrices utriusque Corporis in loco A aequales : Erunt ergo Arcus AP , $A p$ Arcus simul descripti; et Velocitas Corporis P erit ad Velocitatem Corporis p ut AP ad $A p$, vel QP ad Qp . Restituatur jam Ordinata QpP in locum priorem, et manebit etiamnum Ratio QP ad Qp .

In idem Corollarium.

Ex hoc Corollario facile elicitur THEOREMA sequens ; nimirum. Si Vis centripeta ad Centrum quodvis tendens fuerit ut Distantia locorum ab hoc Centro, Corpora omnia, sive propiora sive remotiora ; Temporibus aequalibus e Quiete ad Centrum descendent. Describant enim Circulos vel Ellipses, et omnium Tempora periodica erunt aequalia, quaecunque fuerint Ellipseon Longitudines vel Latitudines. Minuantur Latitudines in infinitum, et descensus obliqui jam vertentur in rectilineos ; et descensuum Tempora omnia aequalia erunt $\frac{1}{4}$ Temporis periodici Corporis circa idem Centrum uniformiter in Circulo revolventis.

In

In SCHOLIUM Prop. 10.

Sunto (in eadem Fig.) AP , Ap duae Lineae curvae communem habentes Verticem A , abscissamque communem AQ ; ea vero sit harum Curvarum indoles, ut acta communi Ordinata QpP , Ordinata QP sit semper ad Ordinatam Qp in data Ratione. Revolvantur jam in his Figuris duo Corpora P et p Vi centripeta tendente ad Punctum quodvis S in abscissa communi AQ eave producta constitutum; sintque Corporum P et p Tempora periodica aequalia: Dico *Vim centripetam in loco P fore ad Vim centripetam in loco p ut Distantia SP ad Distantiam $S p$.*

Primo enim cum Tempora periodica aequentur, Areae simul descriptae erunt ut Areae totae harum Figurarum, hoc est, in Ratione QP ad Qp . Jam vero Figura mixtilinea APQ erit ad Figuram mixtilineam ApQ ut QP ad Qp ; et Triangulum SPQ est ad Triangulum $S p Q$ etiam in Ratione QP ad Qp ; ergo (componendo) erit Area mixtilinea ASP ad Aream ASp ut QP ad Qp : Quare quo tempore Radius SP percurrit Aream ASP , Radius $S p$ percurrent Aream ASp ; adeoque (si Corpora P et p simul exeant de loco A) simul pervenient ad Ordinatam communem QpP . Quare Motus Corporum P et p secundum Rectas ipsi AS parallelas agentes, aequales erunt. Porro cum QP semper sit ad Qp in data Ratione, motus et motuum mutationes, et Vires mutantes secundum

dum Rectas ipsi PQ parallelas agentes, semper erunt in data Ratione QP ad Qp . Referat jam SP Vim centripetam Corporis P in loco quovis P , et resolvi potest Vis SP in Vires PQ, QS ; ergo Vis centripeta Corporis p resolvi potest in Vires pQ, QS : Ergo Vis centripeta Corporis p exprimi debet per Rectam Sp ; quare Vis centripeta in loco P est ad Vim centripetam in loco p , ut Distantia SP ad Distantiam Sp . Q. E. D.

Servatis Ordinatarum QP, Qp Longitudinibus, mutetur jam Angulus SQp , ne Ordinatae QP, Qp amplius jaceant in eadem Recta; et applicari potest praecedens Demonstratio ad hunc casum, similiter atque ad priorem; hoc est, Corpora P et p simul pervenient ad Ordinatas QP, Qp , et Vis centripeta in loco P erit ad Vim centripetam in loco p adhuc ut Distantia SP ad Distantiam Sp . Q. E. D.

Theorema ad Demonstrationem Corollarii primi Propositionis 13. apprime necessarium.

Fig. 69.

Detur Figura quaevis rectilinea $ABCDEFG$, et Punctum quodvis S intra hujus Cavitatem constitutum; et in Latere AB feratur Corpus quacunque cum Velocitate: Dico ita modulari ac temperari posse impulsus centripetos ad Centrum S tendentes, ut Corpus illud in dictae Figurae perimetro perpetuo moveatur.

DEMON-

DEMONSTRATIO.

Acta enim Cc ipsi SB parallela, et occurrente productae AB in c , completoque Parallelogrammo $BcCK$, exponatur Motus Corporis in Latere AB per Rectam Bc , et appulso Corpore ad Angulum B , agat Vis centripeta versus S impulsu unico BK ; et haec Vis BK una cum Vi Corporis insita Bc efficiet ut Corpus deinceps moveatur in Latere BC cum Velocitate quae est ad Velocitatem in AB ut BC ad Bc . Similiter ubi Corpus accenderit ad Angulum proximum C , agat rursus Vis centripeta impulsu simplici versus S , detorquens Corpus de Latere BC in Latus CD , idemque semper fieri intelligatur ad singulos reliquos Angulos D, E, F, G, A , et movebitur Corpus in perimetro Figurae propositae. Q. E. F.

COROLLARIUM.

Valebunt etiam praecedentia in Figuris curvilineis, cum in his Figurae rectilineae inscriptae et circumscriptae ultimo terminentur; adeoque nulla extat Figura curvilinea versus Centrum Virium cava, in cujus perimetro Corpus iuxta Vi centripeta ad Centrum illud tendente moveri non potest, etiamsi Velocitas in loco aliquo particulari sit data, et Mathematicorum utique est harum Virium leges exquirere, et Calculo determinare. Velocitates autem atque adeo Vires centripetae in iisdem locis eadem semper esse debent; siquidem Areae circa Centrum Virium descriptae proportionales sint Temporibus.

Ad

Plate XI.

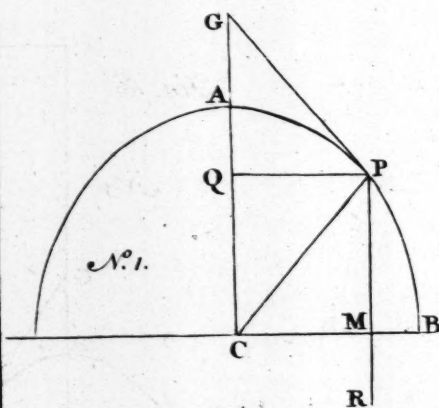
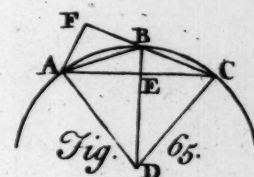
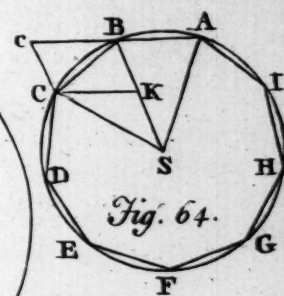
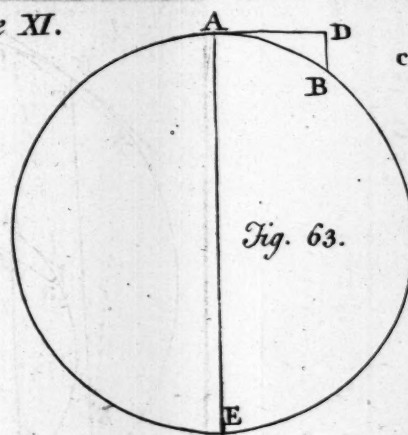
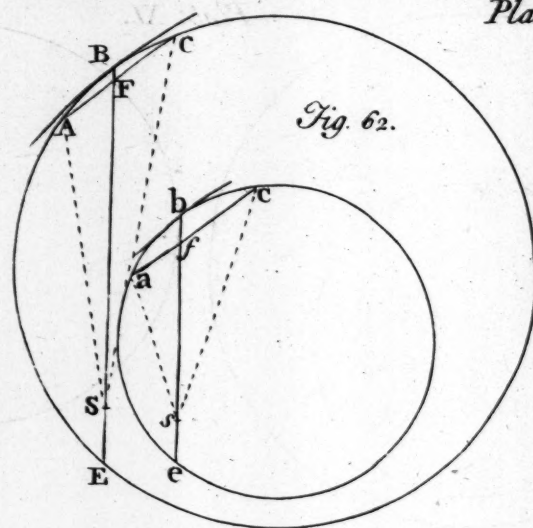
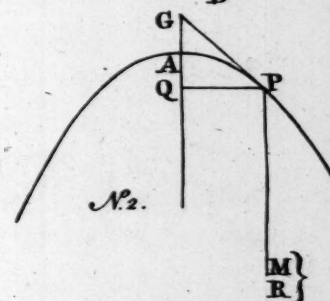


Fig. 66.

N^o 3.



N^o 2.

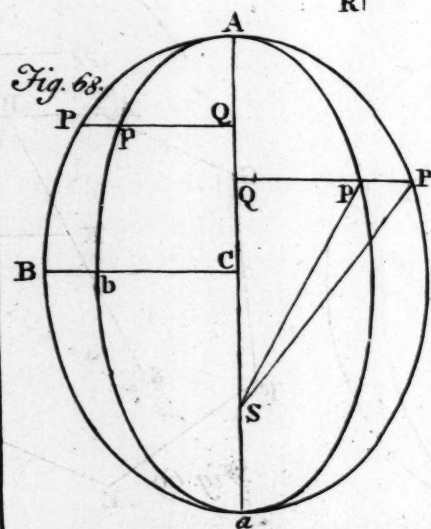


Fig. 68.

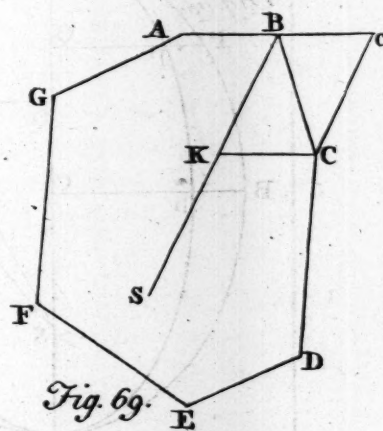


Fig. 69.

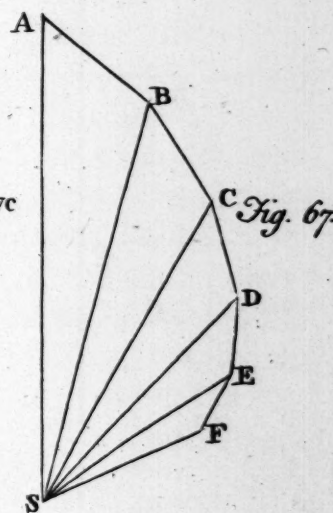


Fig. 70.

Ad Prop. 15.

Sit a Axis major Ellipseos, b minor, l Latus rectum, t Tempus periodicum; et erit $al = b^2$, et $a^3l = a^2b^2$. Est autem ab ut $t\sqrt{l}$, et a^2b^2 ut t^2l : Quare t^2l erit ut a^3l , et t^2 ut a^3 , et t ut $a^{\frac{3}{2}}$. Q.E.D.

Ad Corol. 4. Prop. 16.

Sit t Semiaxis major, c Semiaxis minor, v Velocitas in mediocri Distantia, et exponatur v^2 per Quantitatem $\frac{l}{cc}$; et cum fit $cc = \frac{1}{2}l \times t$, erit $\frac{l}{cc} = \frac{2}{\frac{1}{2}l \times t} = \frac{2}{t}$. Sit jam V Velocitas in Circulo ad Distantiam t , et cum exponatur Velocitas v^2 per $\frac{l}{cc}$, exponetur Velocitas V^2 per $\frac{2t}{tt}$, vel $\frac{2}{t}$. Aequantur itaque V et v .

Ad Corol. 6, ejusdem. Fig. 70.

Sit A a Axis Ellipseos, A Vertex ab Umbilico S remotior, a propior, et Velocitas maxima erit in a , minima in A ; quare ut Velocitatis variatio patefiat, conferamus Velocitatem in a cum Velocitate in A : Est ergo Velocitas in a ad Velocitatem in A ut SA ad Sa ; sed $\sqrt{SA}$ ad $\sqrt{Sa}$ est in minore Ratione quam in Ratione SA ad Sa , propterea quod Ratio prior dimidium tantum fit posterioris; quare Perpendiculara SA et Sa , et proinde etiam Velocitas, magis variantur quam pro reciproca subduplicata Ratione Distantiae.

T

Sit

Sit a Vertex Hyperbolae, Fig. 71. cujus Umbilicus est S , (et non oppositae) et ducatur Sy ad Asymptoton perpendicularis, et Velocitas in a erit ad Velocitatem ad Distantiam infinitam ut Sy ad Sa : Sed Ratio Sy ad Sa finita est, quando Ratio Distantiarum, et proinde etiam subduplicata Ratio, est infinita; quare in Hyperbola, perpendicularum et Velocitas minus variantur quam pro reciproca subduplicata Ratione Distantiae.

Ad Corol. 7.

Sit in iisdem Figuris, Aa Axis major Ellipseos vel Hyperbolae, Bb minor, SA Distantia maxima, Sa minima, L Latus rectum, C Centrum, V Velocitas in Ellipsi vel Hyperbola, v Velocitas in Circulo ad eandem Distantiam, et ostendendum est fore V ad v in minore vel maiore quam subduplicata Ratione 2 ad 1, prout Corpus P in Ellipsi vel Hyperbola versatur. Nam in Ellipseos Vertice A est V ad v in subduplicata Ratione L ad $2SA$ (per hujus Corol. 3.); est autem L ad $2SA$ ut $L \times Aa$ ad $2SA \times Aa$; hoc est, ut $\overline{Bb}^2$ ad $2SA \times Aa$, vel ut $4SA \times Sa$ ad $2SA \times Aa$, vel ut $2Sa$ ad Aa : Quare V est ad v in subduplicata Ratione $2Sa$ ad Aa : Est autem $2Sa$ minor quam $2Ca$, hoc est, quam Aa : Ergo in Vertice A , V minor est quam v ; tantum abest ut major sit in subduplicata Ratione 2 ad 1. Similiter in Vertice a , est V ad v in subduplicata Ratione $2SA$ ad Aa , hoc est, in minore quam subduplicata Ratione $2Aa$ ad Aa vel 2 ad 1.

Pari

Pari Ratiocinio, si sit a Vertex Hyperbolae cujus Umbilicus est S , est V ad v in Vertice a in subduplicata Ratione $2 SA$ ad Aa , hoc est, in majore quam subduplicata Ratione $2 Aa$ ad Aa vel 2 ad 1 . Abeat jam Corpus P ad Distantiam infinitam, et acta Sy ad Asymptoton perpendiculari, est V ad v in subduplicata Ratione $\frac{L}{Sy^2}$ ad $\frac{2SP}{SP^2}$ vel $\frac{2}{SP}$: Sed $\frac{L}{Sy^2}$ est Quantitas finita, et $\frac{2}{SP}$ infinite parva: Quare in Hyperbola ad Distantiam infinitam est V infinities major quam v , et propterea in majore quam subduplicata Ratione 2 ad 1 .

LEMMA.

Si Ellipsin vel Hyperbolam tangat recta quavis, et ex umbilicis demittantur in tangentem perpendiculares, erit sub his perpendicularibus rectangulum aequale quadrato semiaxis conjugati.

Exstat demonstratum ad calcem Operum post-humorum *Cotesii*.

Loco Propositionis 17. Libri I. PRINCIPIORUM, et Loco Corollariorum ad Prop. 16. ejusdem, sequentem adhibe cum Corollariis suis.

PROBLEMA. Figg. 70 et 71.

Exeat Corpus aliquod de loco dato P data cum Velocitate secundum datam positione Rectam PY ,

T_2

et

et simul urgeatur a Vi centripeta tendente ad Centrum datum S, quae in loco quidem P innotescat, aliis vero in locis sit reciproce ut Quadrata Distantiarum ab hoc Centro; *Quaeritur* SECTIO CONICA quam Motu suo descripturum est hoc Corpus.

Cum tangat Recta Z P Y Figuram quaesitam in loco P, et cum detur Positione Recta S P per umbilicorum alterum S transiens, dabitur Positione Recta P H quae per alterum Figuræ umbilicum H transibit, et cujus inventa Longitudo, solvet Problema.

Sunt itaque Velocitas et Vis centripeta in loco P ejusmodi, ut quo Tempore percurritur Longitudo a uniformi cum Velocitate Corporis in P, eodem Tempore, Vi uniformi acceleratrice quae ubique aequalis est Vi centripetae in loco P, recta descenderetur per Altitudinem inter quam et Altitudinem $2 S P$ Longitudo quaedam b media est Proportionalis: Atque his quidem datis, dabitur et Forma et Positione Trajectoria quaesita: Erit enim ut $2 b b - a a$ ad $a a$ ita S P ad P H, inter quas Semiaxis transversus medius est arithmeticus; et si in Tangentem Z P Y demittantur Perpendiculares S Y, H Z, erit etiam ut $2 b^2 - a^2$ ad a^2 ita S Y ad H Z, inter quas Semiaxis conjugatus medius est geometricus.

DEMONSTRATIO.

Nam quo Tempore percurritur Longitudo a , Motu aequabili, vel Longitudo $\frac{b^2}{2 S P}$, Motu aequabiliter accelerato, eodem Tempore Corpus in Circulo

Circulo revolvens ad Altitudinem SP percurreret Arcum b ; uti ex *Corol.* 9, *Prop.* 4, *Lib.* 1, *Principiorum*, manifestum est: Ergo Velocitas in Circulo ad Altitudinem SP est ad Velocitatem in conica Sectione ad eandem Altitudinem ut b ad a , vel in subduplicata Ratione b^2 ad a^2 . Est jam umbilicus alter H, quaecunque tandem fuerit Longitudo PH; et cum Sectiones omnes conicae ad *Ellipsin* referri possunt, ponamus Sectionem quaesitam ellipticam esse; et Quadratum Semiaxis conjugati aequale erit et Rectangulo $\overline{SP+PH} \times \frac{L}{4}$, et Rectangulo SY $\times$ HZ; ergo $\overline{SP+PH} \times \frac{L}{4}$ aequale erit SY $\times$ HZ: Sed ob similia Triangula SPY, HPZ, erit HZ = SY $\times \frac{PH}{SP}$, et SY $\times$ HZ = $\overline{SY}^2 \times \frac{PH}{SP}$; quare est $\overline{SP+PH} \times \frac{L}{4} = \overline{SY}^2 \times \frac{PH}{SP}$; quare $\overline{SP+PH} \times \frac{SP}{PH}$ aequatur $4 \frac{\overline{SY}^2}{L}$. Sed Velocitas in conica Sectione ad Altitudinem quamvis SP, est reciproce in subduplicata Ratione Longitudinis $\frac{\overline{SY}^2}{L}$, vel $4 \frac{\overline{SY}^2}{L}$ per *Prop.* 16, *Lib.* 1, *Principiorum*: Ergo eadem Velocitas est etiam reciproce in subduplicata Ratione Longitudinis $\overline{SP+PH} \times \frac{SP}{PH}$: Ergo Velocitas Corporis in Circulo revolventis ad Altitudinem primam SP est

T 3

ad

ad Velocitatem Corporis e loco suo primo P exe-
 untis, in subduplicata Ratione $\overline{SP+PH} \times \frac{SP}{PH}$ ad
 $\overline{SP+SP} \times \frac{SP}{SP}$, vel in subduplicata Ratione
 $SP+PH$ ad $2PH$. Est ergo b^2 ad a^2 ut
 $SP+PH$ ad $2PH$; $2b^2$ ad a^2 ut $2SP+2PH$
 ad $2PH$; vel ut $SP+PH$ ad PH , et (divi-
 dendo) $2b^2-a^2$ ad a^2 ut SP ad PH vel ut SY ad
 HZ . Q. E. D.

COROL. I.

Orbita descripta *Ellipsis* erit vel *Parabola* vel
Hyperbola, prout Quantitas a^2 minor fuerit, vel
 æqualis, vel major quam $2b^2$.

COROL. II.

Si *Parabola* sit, erit hujus Latus rectum $\frac{4SY^2}{SP}$
 per Lemma 14, et etiam ex Solutione hujus Pro-
 blematis; et Axis per S transiens erit Recta ipsi
 PH parallela.

COROL. III.

Si *Ellipsis* sit vel *Hyperbola*, erit Ellipseos Semi-
 axis $SP \times \frac{b^2}{2b^2-a^2}$, et Semiaxis Hyperbolæ
 $SP \times \frac{b^2}{a^2-2b^2}$. Porro Semiaxis conjugatus in
 casu priore erit $SY \times \sqrt{\frac{a^2}{2b^2-a^2}}$, in posteriore
 $SY \times \sqrt{\frac{a^2}{a^2-2b^2}}$.

COROL.

COROL. IV.

Itaque si detur Velocitas, et Vis centripeta ad Altitudinem quamvis SP, idem erit Axis transversus, quicumque fuerit Angulus SPY ad eandem Altitudinem; et Axis conjugatus erit ut Sinus Anguli istius SPY.

COROL. V.

Si producat SPQ in Ellipsi, vel PSQ in Hyperbola donec SQ aequalis sit Axi transverso Figurae, erit Velocitas in conica Sectione ad Velocitatem in Circulo ad eandem quamvis Altitudinem SP in subduplicata Ratione PQ ad $\frac{1}{2}$ SQ.

COROL. VI.

In Ellipsi, Velocitas Corporis P in mediocri sua ab umbilico Distantia aequatur Velocitati in Circulo ad eandem Distantiam.

COROL. VII.

In Parabola Velocitas est ad Velocitatem in Circulo ad eandem Distantiam in subduplicata Ratione numeri binarii ad unitatem.

COROL. VIII.

Cum Velocitas in Circulo sit ubique reciproce in subduplicata Ratione Distantiae SP, erit Velocitas in conica Sectione in subduplicata Ratione Longitudinis PQ directe, et in subduplicata Rectanguli PSQ reciproce.

COROL. IX.

Adeoque in data Figura, Velocitas erit ubique ut

$$\sqrt{\frac{PQ}{SP}}$$

COROL. X.

Adeoque in data Figura, si detur Velocitas ad Altitudinem quamvis, dabitur illa ad aliam quamvis Altitudinem.

COROL. XI.

In Parabola Velocitas est reciproce in subduplicata Ratione Altitudinis SP.

COROL. XII.

In Hyperbola Velocitas ad Altitudinem quamvis SP est ad Velocitatem in Altitudine infinita in subduplicata Ratione PQ ad PS.

COROL. XIII.

Eadem omnia obtinebunt, siue Latitudo Orbitae descriptae major sit, siue minor, siue nulla.

COROL. XIV.

Adeoque si Velocitates Corporum directe vel oblique ascendentium vel descendentium in aliquo aequalium Altitudinum casu aequales fuerint, et ad omnes alias Altitudines aequales, Velocitates aequales erunt, quaecunque fuerint Latitudines Figurarum descriptarum.

COROL.

COROL. XV.

Si Corpus sursum emittatur de loco P cum Velocitate quae sit ad Velocitatem in Circulo ad eandem Altitudinem, in minori Ratione quam $\sqrt{2}$ ad 1, seu (quod eodem redit) si Velocitas et Vis centripeta in loco P ejusmodi fuerint, ut sit a^2 minor quam $2b^2$, et producat^r Recta SPQ, ita ut sit

$$PQ = \frac{a^2}{2b^2 - a^2} \times SP, \text{ vel } SQ = \frac{2b^2}{2b^2 - a^2} \times SP,$$

erit SQ Altitudo maxima Corporis in Recta illa SPQ ascendenti; et Velocitas ad omnes Altitudines inter ascendendum et descendendum determinabitur per hujus *Corol.* 9 et 10.

COROL. XVI.

Quod si tanta fuerit Velocitas in loco P, ut sit $a^2 = 2b^2$, ascendet Corpus ad Altitudinem infinitam, neque unquam redibit; et Velocitas determinabitur ubique per hujus *Corol.* 11.

COROL. XVII.

Si major adhuc fuerit Velocitas in loco P, Corpus non modo ascendet ad Altitudinem infinitam, sed et Motus sui Partem aliquam etiamtum retinebit. Ut si producat^r Recta PSQ, ita ut sit

$PQ = \frac{a^2}{a^2 - 2b^2} \times SP$, Puncto Q jam existente ad partes alteras Puncti S, Velocitas prima in loco P erit ad Velocitatem ultimam ad Altitudinem infinitam in subduplicata Ratione PQ ad PS, et in omnibus aliis Altitudinibus, Velocitas erit ut in hujus *Corol.* 12.

COROL.

COROL. XVIII.

Si Corpus de loco quovis Q demissum descendat usque ad Centrum S , Velocitas ad Altitudinem quamvis minorem SP est ad Velocitatem Corporis in Circulo revolventis ad eandem Altitudinem in subduplicata Ratione PQ ad $\frac{1}{2}SQ$.

COROL. XIX.

Et Tempus totius descensus, dimidium erit Temporis periodici Corporis in Circulo ad Altitudinem $\frac{1}{2}SQ$ revolventis. Nam spectandus est hic descensus tanquam factus in Semiperimetro Ellipseos nullius Latitudinis, cujus tamen Longitudo seu Axis transversus est SQ .

COROL. XX.

Ergo Tempus totius descensus erit in sesquuplicata Ratione Altitudinis totius SQ .

SCHOLIUM.

Posteaquam haec conscripsissem, incidi in aliam praecedentis Problematis Constructionem paulo simpliciore quam hic subungere visum est.

I. Sunto Velocitas et Vis centripeta in loco P ejusmodi, ut quo Tempore percurritur Longitudo a uniformi cum Velocitate Corporis in P , eodem Tempore Vi uniformi acceleratrice quae ubique aqualis est Vi centripetae in loco P , descenderetur per Spatium s , fiatque ipsis s et a tertia proportionalis $4l$; eritque SP ad PH , vel SY ad HZ ut
 $SP—l$

$SP-l$ ad l ; Nam Velocitas Corporis in Circulo revolventis ad Altitudinem SP est ad Velocitatem in conica Sectione ad eandem Altitudinem ut media Proportionalis inter $2SP$ et s , ad a , five mediam Proportionalem inter $4l$ et s , hoc est, in subduplicata Ratione SP ad $2l$. Sed in Demonstratione praecedentis Solutionis, Fig. 70, erat Velocitas illa ad hanc in subduplicata Ratione $SP+PH$ ad $2PH$; quare $SP+PH$ est ad $2PH$ ut SP ad $2l$; unde erit SP ad PH vel SY ad HZ ut $SP-l$ ad l . Q. E. D.

Longitudo Axis transversi est $\frac{\overline{SP}^2}{SP-l}$, Axis conjugati $2SY \times \sqrt{\frac{l}{SP-l}}$, et Lateris recti (quod Axis tertium est Proportionale) $4l \times \frac{\overline{SY}^2}{\overline{SP}^2}$.

Porro si Longitudo a nunc major, nunc minor accipitur, erit Spatium s ut Quadratum Temporis quo Longitudo illa percurritur; adeoque erit s ut a^2 : Dabitur itaque Quantitas $\frac{a^2}{s}$ vel $4l$, utcunque magna vel exigua sumatur Longitudo a ; et propterea quaecunque fuerit lex vel Quantitas Vis centripetae aliis in locis, si in Recta PVS vel PSV capiatur Longitudo PV aequalis Longitudini $4l$ ad locum P pertinenti, Circulus qui tangit Rectam PY in P transique per V , Curvaturam habebit Trajectoriae descriptae in loco illo P , et Diameter Curvaturae major erit quam Chorda PV in Ratione SP ad SY . (Vide *Prop. 7, Lib. 1, Prin-*

Principiorum) Et si fumatur Arcus quam minimu. Pq in consequentia, Angulus SPq perpetuo augebitur vel minuetur, prout in Recta PSV Longitudo SV major fuerit vel minor quam SP , vel PV quam $2PS$, vel prout Velocitas in loco P major fuerit vel minor quam quae sufficiat ad Circulum describendum ad Altitudinem SP , quando Angulus SPQ rectus est.

Consequuntur haec ex natura Circuli, pendet itaque Incrementum vel Decrementum Anguli SPq ex Altitudine SP , et ex Velocitate et Vi centripeta in loco P ; accessus autem Corporis ad Centrum S , vel recessus ab eodem, ab his minime dependent, sed ex sola Quantitate Anguli SPq : Accedit enim vel recedit, prout Angulus iste minor est vel major Recto.

2. Diximus in hoc Scholio augeri vel diminui Angulum SPq in consequentia, prout Longitudo PV major fuerit vel minor quam $2PS$, vel prout Velocitas in loco P major fuerit vel minor quam Velocitas Corporis ad eandem distantiam SP eademque Vi centripeta in P Circulum describentis: Nam Angulus SPq aequalis est Angulo in alterno segmento per *Prop. 32, Lib. 3, Elem.* et proinde aequalis est Angulo ad Centrum cuius Mensura semissis est Arcus PqV . Augetur itaque vel diminuitur Angulus SPq pro Ratione aucti vel diminuti Arcus illius.

Feratur Corpus P , Fig. 72, in Orbe circulari $ABCD$, existente Centro S extra Centrum Circuli

et

et percurrente Corpore P Arcum C D E F G, Arcus P q V, et Angulus S P q perpetuo augetur, et Longitudo P V major est Longitudine 2 P S per *Prop.* 7 et 8, *Lib.* 3, *Elem.* Percurrente vero Corpore Arcum G H A B C, Arcus P q V, et Angulus S P q perpetuo diminuitur, et Longitudo P V minor est quam 2 P S: par est Ratio in Arcu quolibet Aequo-curvo. Porro, quo Tempore Corpus P, urgente uniformi Vi centripeta in P descenderet per Longitudinem s , idem eodem Tempore cum Velocitate sua in P uniformiter percurreret Longitudinem a , quae ex Hypothesi media est proportionalis inter $4 l$ et s ; et eodem etiam Tempore Corpus ad distantiam S P, eadem Vi centripeta in Circulo revolutum percurreret Arcum qui medius est proportionalis inter Diametrum 2 P S et Spatium s . Est ergo Velocitas in P ad Velocitatem in Circulo ad distantiam S P in subduplicata Ratione $4 l \times s$ ad $2 P S \times s$, vel $4 l$ ad 2 P S, vel P V ad 2 P S. Est ergo Velocitas in P major vel minor quam Velocitas in Circulo ad distantiam S P, prout Longitudo P V major est vel minor quam Longitudo 2 P S.

PROBLEMA. Fig. 70.

Sit S Centrum Terrae, S P Altitudo 10 Semi-diametrorum terrestrium, S P Y Angulus, cujus Sinus est ad Radium ut 3 ad 4, et emittatur Corpus de loco P secundum directionem P Y, cum Velocitate qua uniformiter percurri possunt pedes Parisienses $4865 \frac{11}{71}$ Tempore unius Minuti secundi.

Quaeritur

Quaeritur Forma et Positio Trajectoriae describendae, et, si Ellipsis fuerit, Tempus periodicum.

SOLUTIO.

1. Pone $4865\frac{11}{71} = a$, et erit $a^2 = 23669732$; et cum ad Superficiem telluris nostrae descendant Corpora omnia per Altitudinem pedum Parisiensium $15\frac{1}{12}$ Tempore unius Minuti secundi, descendunt eadem ad Altitudinem SP per partes $\frac{181}{1200}$ hujus pedis; unde erit $s = \frac{181}{1200}$ et $\frac{a^2}{s}$ vel $4l = 156926400$, et $l = 39231600$.

2. Semidiameter terrestris est pedum Parisiensium 19615800; unde erit SP 196158000 eorundem pedum: Est itaque SP major quam l , et Sectio conica describenda erit *Ellipsis*.

3. Est SP ad l ut 5 ad 1, et SP— l , ad l , vel SP ad PH vel SY ad HZ ut 4 ad 1: Est itaque $PH = 2\frac{1}{2}$ Semidiametris terrestribus, et $SP - PH$ five Axis major Ellipseos aequalis $12\frac{1}{2}$ Semidiametris.

4. Longitudo perpendiculari SY est $7\frac{1}{2}$ Semidiametrorum, et Longitudo perpendiculari HZ $1\frac{7}{8}$ Semidiametrorum; unde est SY $\times$ HZ seu Quadratum Semiaxis minoris $\frac{225}{16}$, et Axis minor erit $7\frac{1}{2}$ Semidiametrorum terrestrium; unde Longitudo Ellipseos erit ad Latitudinem ut 5 ad 3.

5. Si a Quadrato Axis majoris $\frac{625}{4}$ subducatur Quadratum Axis minoris $\frac{225}{4}$, restabit Quadratum Longitudinis

Longitudinis $SH \frac{400}{4}$ vel 100; unde erit $SH = 10$ Semidiametris, et hujus dimidium 5 est distantia umbilicorum a Centro Ellipseos: Est itaque distantia Apogaei a Centro Terrae $11\frac{1}{4}$ Semidiametrorum terrestrium, et distantia Perigaei $1\frac{1}{4}$.

6. Cum dentur omnia Latera Trianguli SPH , invenietur Angulus PSH $14^{\circ} 22'$; unde dabitur positio Lineae *Apsidum* in antecedentia, vel in consequentia, constituendae, prout Angulus SPY est minor vel major Recto.

7. Jam vero ut inveniatur Tempus periodicum in hac Orbita, animadvertendum, Corporis juxta Terrae Superficiem in Circulo revolventis Tempus periodicum inventum fuisse alibi unius horae, 24 minutorum primorum, 27 secundorum, hoc est, 5067 secundorum. Dicendum itaque ut Cubus Diametri Orbitae circularis ad Cubum Axis transversae Ellipseos, ita Quadratum Temporis periodici prioris ad Quadratum Temporis periodici posterioris. Sunt autem dicti Cubi ut 8 ad $\frac{25 \times 25 \times 25}{8}$, vel ut 64 ad $25 \times 25 \times 25$, et horum numerorum Radices quadraticae sunt ut 8 ad $5 \times 5 \times 5$ vel ut 8 ad 125. Dic itaque ut 8 ad 125 ita 5067 minuta secunda, sive Tempus periodicum in Circulo ad 79172 minuta secunda quae constituunt Tempus periodicum quaesitum. Absolvit igitur novus hic Planeta periodum suam circa Centrum telluris Spatio horarum 21, minutorum primorum 59, secundorum 32. Q. E. I.

In Prop. 44. Vide Fig. apud auctorem.

*Centro C, Intervallo Cn describatur Circulus secans
Lineam mn productam in t.*

Quamvis Recta rm infinita sit, tamen Angulus rCm nunquam major erit Recto; adeoque Recta rm erit ad Rectam rk in majori Ratione quam est Angulus rCm ad Angulum rCk : Est autem rm ad rk ut Angulus rCn ad Angulum rCk ; quare Angulus rCn major est Angulo rCm . Sed quo propius Punctum m accedit ad Punctum r , eo minus erit Discrimen Angulorum rCm , rCn ; coeuntibus autem Punctis m et r , aequales erunt Anguli rCm , rCn , hoc est, Punctum n jam situm erit in Recta Cm ; id adeo, propterea quod in hoc casu est rm ad rk ut Angulus rCk , vel rCn ad rCk .

In Corol. 2. ejusdem Prop.

Vis autem qua Corpus in Circulo, &c.

Adhibita Notatione *Newtoniana*, positoque insuper quod Velocitas Corporis P in Vertice sit c , sic argumentor. Vis centripeta Corporis P in Vertice V exponitur per $\frac{F^2}{CV^2}$: Sed Curvatura Ellipseos in vertice V eadem est cum Curvatura Circuli cujus Semidiameter est R , dimidium nimirum Lateris recti, ut patet ex *Conicis*: Ergo si Corpus Circulum describat ad Distantiam R cum Velocitate c , erit Vis centripetae in hoc Circulo
aequalis

aequalis Vi centripetae in Vertice V; ergo Vis centripeta in hoc Circulo erit $\frac{F^2}{C V^2}$. Revolvatur jam Corpus aliquod cum Velocitate c in Circulo ad Altitudinem CV, et Vis centripeta in priore Circulo erit ad Vim centripetam in Circulo posteriore ut $\frac{C^2}{R}$ ad $\frac{C^2}{C V}$ per *Corol. 1, Prop. 4, Lib. 1, Principiorum*; hoc est, ut $\frac{1}{R}$ ad $\frac{1}{C V}$: Dic ergo, ut $\frac{1}{R}$ ad $\frac{1}{C V}$ ita $\frac{F^2}{C V^2}$ ad $\frac{R F^2}{C V^3}$; et $\frac{R F^2}{C V^3}$ erit Vis centripeta Corporis Circulum describentis cum Velocitate c ad Distantiam CV.

Lemma ad Prop. 45.

Si Quantitates duae $a+x$ et $b+y$ componantur ex partibus datis a et b , et ex partibus non datis, simul tamen nascentibus, vel simul evanescentibus x et y ; fuerit autem $a+x$ ut $b+y$; dico fore x ad y semper ut a ad b . Etenim cum ponatur $a+x$ ut $b+y$, erit $a+x$ ad $b+y$ semper in data Ratione: Nascentibus autem vel evanescentibus x et y , est $a+x$ ad $b+y$ ut a ad b ; quare $a+x$ erit ad $b+y$ semper ut a ad b ; adeoque x erit ad y semper ut a ad b . Q. E. D.

Ad Prop. 45.

Ut in Corollariis secundo et tertio.] Sit Vis centripeta ut Altitudinis Dignitas quaelibet, cujus Index est $n-3$: Quaeritur Angulus inter Apfidem summam et Apfidem imam, secundum Hypothesim Corollarii tertii, ubi nimirum Centrum Virium coincidit cum Centro communi Ellipseon.

Secundum hanc Hypothesim, adhibita Notatione *Newtoniana*, Vis centripeta est ut $\frac{F^2 A}{T^3} + \frac{RG^2 - RF^2}{A^3}$.

Scribatur 1 pro T vel R, et Vis centripeta jam erit ut $F^2 A + \frac{G^2 - F^2}{A^3}$, hoc est ut $\frac{F^2 A^4 + G^2 - F^2}{A^3}$:

Est autem $A^4 = 1 - 4X$, &c. et $F^2 A^4 = F^2 - 4F^2 X$,

&c. quare Vis centripeta est ut $\frac{F^2 - 4F^2 X + G^2 - F^2}{A^3}$,

vel ut $\frac{G^2 - 4F^2 X}{A^3}$; ponitur autem Vis centripeta

ut A^{n-3} vel $\frac{A^n}{A^3}$; quare A^n est ut $G^2 - 4F^2 X$:

Sed $A^n = 1 - nX$, &c. quare $G^2 - 4F^2 X$ est ut $1 - nX$; unde est G^2 ad 1 ut $4F^2$ ad n ; unde

$G^2 \times n = 4F^2$, et $G^2 = \frac{4F^2}{n}$, et $G = \frac{2F}{\sqrt{n}}$. Sed in hac

Hypothesi est F 90 Grad. cum sit 90 Grad. Angulus inter Apfidem summam et Apfidem imam in Ellipsi quiescente: Quare Angulus inter Apfidem imam, ubi Vis centripeta est ut A^{n-3} , erit

$\frac{180}{\sqrt{n}}$, idem nimirum qui prodit ex Hypothesi

Corollarii secundi.

Ad

Ad Corol. 1. Prop. 45.

Quantitas $\frac{nn}{mm} - 3$ exponens dignitatem Altitudinis est in casu proposito $\frac{14400}{14641} - 3$: Est etiam $\frac{14641}{14641} = 1$, et si subducatur utrobique $\frac{241}{14641}$, manebit $\frac{14400}{14641} = 1 - \frac{241}{14641}$, et $\frac{nn}{mm} - 3 = 1 - \frac{241}{14641} - 3 = -2 - \frac{241}{14641}$; ergo Vis centripeta est reciproce ut Altitudinis dignitas $2 + \frac{241}{14641}$. Fractionis $\frac{241}{14641}$ divide tum Numeratorem tum Denominatorem per 241, et prodibit $\frac{1}{60 + \frac{1}{\frac{1}{2} + \frac{1}{4}}}$: Sed $\frac{1}{\frac{1}{2} + \frac{1}{4}}$ non multum abest ab $\frac{1}{\frac{3}{4}}$ vel $\frac{4}{3}$; quare Vis centripeta est reciproce ut Altitudinis dignitas $2 + \frac{1}{60 + \frac{1}{\frac{3}{4}}}$, cujus ab Indice 2 differentia est $\frac{1}{60 + \frac{1}{\frac{3}{4}}}$: Sin $2 + \frac{1}{60 + \frac{1}{\frac{3}{4}}}$ subducatur de 3, vel de $2 + \frac{60\frac{3}{4}}{60\frac{3}{4}}$, manebit $\frac{59\frac{3}{4}}{60\frac{3}{4}}$; quare differentia posterior est ad priorem ut $59\frac{3}{4}$ ad 1.

Ad Prop. 58.

Si Velocitates in P et p fuerint in subduplicata Ratione CP ad sp, Corpora P et p describent Arcus similes in Temporibus quae erunt etiam in

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sub-

subduplicata illa Ratione; et si Velocitates in Q et q fuerint ut $\sqrt{CQ}$ ad $\sqrt{sq}$, vel ob similitudinem Arcuum ut $\sqrt{CP}$ ad $\sqrt{sp}$, Corpora P et p pergent Arcus similes Temporibus proportionalibus describere. Videamus ergo an Velocitates in Q et q sint ad invicem in hac Ratione. Anguli CPQ et spq , ob similitudinem situs Corporum P et p in Curvis similibus, aequantur; ergo Vires centripetae, quibus haec Corpora in Lineas curvas detorquentur, similiter applicantur: Sed Vires aequales similiter applicatae, Velocitates generant Temporibus proportionales; ergo Incrementa vel Decrementa Velocitatum in locis Q et q sunt ut $\sqrt{CP}$ ad $\sqrt{sp}$: Addantur haec Incrementa vel subducantur Decrementa Velocitatibus in P et p quae sunt etiam ut $\sqrt{CP}$ ad $\sqrt{sp}$, et prodibunt Velocitates in Q et q etiam in eadem illa Ratione. Sunt ergo Velocitates in locis omnibus similibus ut $\sqrt{CP}$ ad $\sqrt{sp}$.

Ad Prop. 60.

Minuatur Tempus perïodicum in Ratione $\sqrt{S+P}$ ad $\sqrt{S}$; et simul Axis major Ellipseos descriptae in Ratione A ad a , et erit $\sqrt{S+P}$ ad $\sqrt{S}$ ut $A^{\frac{3}{2}}$ ad $a^{\frac{3}{2}}$ (nam Tempora periodica sunt in sesquiplicata Ratione Axium A et a); ergo erit $S+P$ ad S ut A^3 ad a^3 . Sunt m et n mediae proportionales inter $S+P$ et S , ita ut sint $S+P$, m , n , S , continue proportionales, et Ratio $S+P$ ad S aequalis erit Rationi $S+P$ ad m triplicatae, hoc est, erit $S+P$ ad S ut $\overline{S+P}^3$ ad m^3 : Erat autem

autem $S \vdash P$ ad S et A^3 ad a^3 ; quare A^3 erit ad a^3 ut $\overline{S \vdash P}^3$ ad m^3 , adeoque A ad a ut $S \vdash P$ ad m .
Q. E. D.

Ad Prop. 61. partem alteram.

Moveantur duo Corpora S et P circa commune Gravitatis Centrum C , et nominetur SP , x , et CP , z ; et ponantur Corpora S et P aequalia, ita ut sit $x = 2z$; exponamus denique Vim centripetam qua Corpus S trahit Corpus P per Quantitatem aliquam compositam, qualis est $Ax \vdash Bx^2$, habitis A et B pro Quantitatibus quibuscumque datis, et si exponatur lex $Ax \vdash Bx^2$ per potestates ipsius z , evadet $2Az \vdash 4Bz^2$; vel positis a et b pro $2A$ et $4B$, lex Vis centripetae ad commune Centrum C tendentis fiet $az \vdash bz^2$; et Quantitas $az \vdash bz$ semper aequalis erit Quantitati $Ax \vdash Bx^2$. Statuatur jam Corpus aliquod in communi Centro C , quod trahat Corpus P Vi centripeta $az \vdash bz^2$, Corpore S jam destructo; et Corpus P jam trahetur ad Corpus in Centro C constitutum omnino pariter ac prius, tum quoad directionem, tum quoad Quantitatem Vis centripetae.

Ad Prop. 64.

LEMMA.

Si feratur Corpus aliquod L circa Centrum D cum Vi centripeta quae sit ut $A \times DL$, Tempus periodicum erit reciproce in subduplicata Ratione Quantitatis A . Nam (per *Corol. 2. Prop. 10.*)

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Tempus

Tempus periodicum idem erit ac si Corpus L Circulum describeret circa Centrum D ad Distantiam quamvis DL: Describat ergo; et si Tempus periodicum vocetur T, erit $A \times DL$ ut $\frac{DL}{T^2}$ (per *Corol. 2. Prop. 4*); unde erit T^2 ut $\frac{1}{A}$ et T ut $\frac{1}{\sqrt{A}}$. Q. E. D.

Positis quae in *Prop. 64*, trahat Corpus unumquodque Corpora reliqua cum Vi acceleratrice, quae sit ut Corpus trahens et distantia Corporum attractorum conjunctim; et exponetur Vis qua Corpus T trahit Corpus L per Quantitatem $T \times TL$; cumque sit L ad T ut TD ad DL, erit (*componendo*) $T + L$ ad T ut TL ad DL; unde $T \times TL$ aequalis erit $\overline{T+L} \times DL$: Trahitur ergo Corpus L ad Centrum commune D, perinde ac si Corpus T tolleretur, et aliud Corpus $T+L$ in communi Centro D immotum constitueretur; et Tempus periodicum Corporis L circum D erit ad Tempus periodicum Corporis L circum T immotum in subduplicata Ratione Corporis T ad summam Corporum $T+L$. Accedat jam Corpus tertium S, et Vis qua Corpus S trahit Corpus T exponetur per $S \times ST$ aequalis Viribus $\overline{S \times TD} + \overline{S \times SD}$: Similiter Vis qua Corpus S trahit Corpus L, aequalis erit Viribus $\overline{S \times LD} + \overline{S \times SD}$: Quare summa Virium qua Corpus S trahit Corpora T et L est $\overline{S \times TD} + \overline{S \times DL} + \overline{S \times 2SD}$; Sed Vis $S \times 2SD$ minime perturbat Motum Systematis; ut

ut demonstrat *Neutonus*; et Vi $\overline{T+L} \times D L$ qua Corpus L trahebatur ad Centrum D ante Accessum Corporis S, jam additur Vis $S \times D L$, ita ut Vis tota qua Corpus L trahitur ad Centrum D jam sit $\overline{S+T+L} \times D L$; minuitur ergo Tempus periodicum Corporum T et L circa commune ipsorum Gravitatis Centrum D per Accessum Corporis S in subduplicata Ratione $S+T+L$ ad $T+L$; et Corpus alterutrum puta L trahitur ad Centrum D perinde ac si Corpora S et T submoverentur, et Corpus $S+T+L$ in communi Centro D collocaretur. Similiter si accedat Corpus quartum V, trahetur Corpus L ad Centrum D, perinde ac si Corpora V, S, T submoverentur, et Corpus $V+S+T+L$ in Centro D collocaretur, et Tempus periodicum Accessu Corporis V minuetur in subduplicata Ratione $V+S+T+L$ ad $S+T+L$.

Haec sunt Phaenomena motuum binorum quorumcunque Corporum circa commune ipsorum Gravitatis Centrum: Videamus jam Phaenomena motuum respectu reliquorum Centrorum C et B. Corpus S trahitur ad Corpus T Vi centripeta $T \times T S$ aequali Viribus $\overline{T \times T D} + \overline{T \times S D}$; similiter Vis qua Corpus S trahitur ad Corpus L aequatur Viribus $\overline{L \times L D} + \overline{L \times S D}$; quare summa Virium quibus Corpus S trahitur a Corporibus T et L, est $\overline{T \times T D} + \overline{L \times L D} + \overline{T+L \times S D}$. Sed Vires $\overline{T \times T D}$ et $\overline{L \times L D}$ aequales cum sint et contrariae, ex Natura Centri Gravitatis se mutuo

destruunt; manet itaque Vis $\overline{T+L} \times SD$ qua Corpus S trahitur a Corporibus T et L, pariter ac si Corpora illa T et L in unum coalescerent, et in communi Centro D locarentur. Porro cum sint ex Natura Centri Gravitatis S ad T+L ut CD ad CS, et componendo S+T+L ad T+L ut SD ad SC, erit $\overline{T+L} \times SD = \overline{S+T+L} \times SC$; ergo Corpus S trahitur ad commune Centrum C cum Vi $\overline{S+T+L} \times SC$. Accedat rursus Corpus quartum V, et Vis qua V trahitur ad S erit $S \times VS = \overline{S} \times SC + \overline{S} \times VC$; et Vis qua Corpus V trahitur a Corporibus T et L erit $\overline{T+L} \times CD + \overline{T+L} \times VC$ uti ante dictum est; quare Vis tota qua Corpus V trahitur a Corporibus S, T, L, est $\overline{S} \times SC + \overline{T+L} \times CD + \overline{S+T+L} \times VC$: Sed Vires $S \times SC$ et $\overline{T+L} \times CD$ se mutuo destruunt, ut prius; quare Vis qua Corpus V trahitur a Corporibus S, T, L est $\overline{S+T+L} \times VC$: Est autem V ad S+T+L ut BC ad VB, et componendo, $V+S+T+L$ est ad S+T+L ut VC ad VB: Est ergo $\overline{S+T+L} \times VC = \overline{V+S+T+L} \times VB$. Trahitur ergo Corpus unumquodque V ad commune omnium Centrum B omnino similiiter ac si Corpus $V+S+T+L$ aequale nempe toti Systemati, in communi Centro B constitueretur; et Vires absolutae ad Centra B, C, D, tendentes aequales sunt inter se. Sunt ergo Tempora periodica tum Corporis unius cujusque circa Centrum B, tum trium quorumcunque circa Centrum C, tum

tum duorum quorumcunque circa Centrum D, aequalia inter se; et ad Tempus periodicum Corporis L circa Centrum immotum T in subduplicata Ratione Corporis T ad totum Systema $V+T+L$. Trahant jam Corpora T et L se mutuo Viribus vel majoribus vel minoribus quam quibus trahunt caetera, et Tempus periodicum Systematis $T+L$ circa Centrum D minuetur vel augebitur in subduplicata Ratione Virium auctarum vel diminutarum; sed Motus reliquorum ex hac Virium intensiōe vel remissione minime perturbabitur.

Prop. 66. Corol. 6. Fig. 73.

Revolvatur Corpus P circum T Tempore a , existente mediocri distantia $TP = d$; dein per actionem Corporis S minuatur Vis centralis Corporis T in Ratione r ad s , et per hanc diminutionem efficietur ut Corpus P jam revolvatur circum T ad majorem Distantiam; quam Distantiam in mediocri sua Quantitate vocabimus D, et majori etiam Tempore periodico quod sit c ; quaeritur Ratio inter a et c .

Fingamus aliud Corpus circum T revolvens ad Distantiam D Tempore b , inter quod et Corpus S nulla intercedit attractio: quo posito erit Tempus a ad Tempus b ut $d^{\frac{3}{2}}$ ad $D^{\frac{3}{2}}$. Sed per *Prop. 4.*

Corol. 2. $\frac{D}{b^2}$ est ad $\frac{D}{c^2}$ ut r ad s ; adeoque b erit ad c ut $\sqrt{s}$ ad $\sqrt{r}$. Componantur hae Rationes, et prodibit a ad c ut $d^{\frac{1}{2}} \times \sqrt{s}$ ad $D^{\frac{1}{2}} \times \sqrt{r}$.

In

In Corol. 10 et 11.

Orbitam PAB Linea Nodorum in duos Semicirculos dirimit, quorum uterque binas habet facies, alteras plano EST inclinatas, alteras reclinatas; et quanquam Vis MN agat semper secundum rectas plano EST parallelas, dicemus tamen in sequentibus agere eam *ad* planum vel *à* plano EST , prout ab inclinata vel reclinata facie immediate dirigatur. Dicemus etiam motum Corporis P fieri *ad* planum vel *à* plano EST , prout Corpus illud *ad* Nodum proximum vel *à* Nodo proximo tendit. His praemissis Effata quaedam proferemus, quibus ad Corollaria supra memorata aditus amplissimus pateat, quae alias vix aut ne vix quidem intelligi possunt. Horum vero Demonstrationes ad *Machinulam* quandam simplicissimam, et huic potissimum Negotio constructam, referemus.

N. B. Constat haec Machina ex duobus annulis circularibus plano cuidam planum EST referenti inclinatis, et adglutinis, et se mutuo extra hoc planum in Angulo quam fieri potest minimo decussantibus.

1. Si vis MN et Motus Corporis P ejusdem fuerint affectionis, hoc est, si fiat uterque *ad* planum vel *à* plano EST , inclinatio Orbitae PAB , hoc est, Angulus inclinationis perpetuo augebitur: Alias minuetur.

2. Vis MN , et motus Corporis P , ejusdem sunt affectionis in locis Orbitae PAB oppositis, et propterea

propterea quicquid praestant in uno Semicirculo, idem praestabunt in altero.

3. Linea Quadraturarum fertur in consequentia eadem Velocitate cum Linea Syzygiarum, idque sive motus iste motui Corporis S circum T vel Corporis T circum S debeatur.

4. Inter Quadraturam et Nodum proximum, Vis MN agit a plano EST, aliis in locis ad planum.

5. Si Vis MN agat ad planum EST, Nodi regrediuntur, alias progrediuntur.

In Corol. 14.

Quo melius intelligantur ea quae in hoc Corollaria tradita sunt, Longitudines LM, MN quibus Vires omnes perturbatrices Corporis P exponuntur, primo sunt investigandae. Nam si detur Distantia ST, accipi potest Longitudo SK aequalis ipsi ST; quo in casu erit etiam LM in mediocri sua Quantitate aequalis Distantiae PT; sin augeatur Distantia ST, necesse erit ut minuatur Longitudo SK; adeoque neque Longitudines SK, ST, neque Longitudines LM, PT amplius pro aequalibus habendae sunt, sed eruenda est Longitudo LM ex similibus Triangulis SLM, SPT per sequentem Analogiam, nempe SP est ad PT ut SL ad LM, adeoque $LM = \frac{SL \times PT}{SP}$:

Verum quoniam SL nunc major est, nunc minor quam SK, et ab eadem SK nunquam aberrat sensibilibiter, exponatur SL per mediocrem suam

Quan-

Quantitatem SK, et similiter SP per ST, et prodibit Longitudo LM = $\frac{SK \times PT}{ST}$.

Jam vero ut exquiramus Longitudinem MN, jungatur KN, et ad Rectam ST perpendiculares ducantur LX, PI, et si Orbita PAB circularis sit, vel propemodum circularis, aequales erunt SK, SN, et Anguli SKN, SNK tantum non erunt recti propter Angulum ad S tantum non evanescentem: Quare Rectae KN et LX pro parallelis habendae sunt ut et Rectae KL, NX pro aequalibus et parallelis: Sed et Rectae SP, SI erunt etiam aequales ob Angulum ad I rectum; quare cum sit SK ad SL ut $\overline{SP}^2$ ad $\overline{ST}^2$, erit etiam SK ad SL ut $\overline{SI}^2$ ad $\overline{ST}^2$, vel ut SI ad $SI + 2IT$, et dividendo SK erit ad KL ut SI ad $2IT$: Prodit ergo KL vel NX = $\frac{SK \times 2IT}{SI} = \frac{SK \times 2IT}{ST}$. Rursum, propter similia Triangula SLM et SPT, LMX et PTI, erit MX ad IT ut LM ad PT, hoc est, ut SL ad SP; quare MX aequatur $\frac{SL \times IT}{SP} = \frac{SK \times IT}{ST}$; quare Longitudo MN feu $MX + XN$ erit $\frac{SK \times 3IT}{ST}$. Et universaliter LM erit ad MN ut Radius ad Triplum Cofinus Anguli PTS; et in dato situ Corporum S, T, P, ad invicem, ubi est $3IT$ ut PT, ob datum Specie Triangulum PTI, erit MN

MN ut $SK \times \frac{PT}{ST}$; quare Vis LM universaliter, et Vis MN in integra Revolutione Corporis P circum T, est ut $SK \times \frac{PT}{ST}$.

His expeditis, fit V Vis absoluta attractiva Corporis S, et SK erit ut $\frac{V}{ST^2}$; ergo $SK \times \frac{PT}{ST}$, seu Vires LM, MN erunt ut $\frac{V \times PT}{ST^3}$; unde in dato Systemate Corporum P, T, erunt Vires LM, MN ut $\frac{V}{ST^3}$. Sit jam T Tempus periodicum Corporis T circum S, et per *Corol. 2. Prop. 4.* hujus, erit $\frac{V}{ST^2}$ ut $\frac{ST}{T^2}$, et $\frac{V}{ST^3}$ ut $\frac{1}{T^2}$; ergo in dato Systemate Corporum P, T, Vires LM, MN sunt reciproce ut Quadratum Temporis periodici Systematis circum S revolventis. Postremo, fit D Diameter Corporis S, et Diameter ejus apparens ex Centro T spectata erit ut $\frac{D}{ST}$, et Cubus Diametri apparentis ut $\frac{D^3}{ST^3}$. Sunt V et D^3 semper proportionales, hoc est, Magnitudo Corporis S et Vis absoluta attractiva vel maneant vel mutantur proportionaliter, eritque $\frac{V}{ST^3}$ ut $\frac{D^3}{ST^3}$, hoc est, Vires LM, MN erunt ut Cubus Diametri apparentis Corporis S ex Centro T spectatae.

In

In Corol. 15.

Sit V Vis attractiva Corporis S , v Vis attractiva Corporis T , D Diameter Orbitae Corporis T circum S revolventis, d Diameter Orbitae PAB , T Tempus periodicum Corporis T circum S , t Tempus periodicum Corporis P circum T , et per *Corol. 2. Prop. 4.* erit $\frac{D}{T^2}$ ut V , et $\frac{D}{V}$ ut T^2 ; unde

erit T^2 ad t^2 ut $\frac{D}{V}$ ad $\frac{d}{v}$, vel ut Dv ad Vd , hoc est;

T^2 erit ad t^2 in Ratione composita ex Ratione D ad d et Ratione v ad V . Manente Orbium proportionem mutantur eorum Magnitudines, et vel maneant vel mutantur proportionaliter Vires V et v , et propter servatas Rationes componentes inter D et d , et v et V , servabitur Ratio composita inter T^2 et t^2 , eritque T ad t in eadem Ratione qua prius. Tangat jam Recta PR Orbitam PAB in loco P , et a loco proximo Q age QR Distantiae PT parallelam, et quo Tempore Corpus P sola Vi infita percurreret Tangentem PR , vel addita Vi v percurreret Arcum quam minimum PQ , eodem Tempore idem P totis Viribus LM , MN et v percurrat Arcum Px , et Lineola Qx erit Effectus Virium LM , MN .

Manentibus Orbium Forma, Proportionem, et Inclinationem ad invicem, mutantur eorum Magnitudines, ut et Magnitudo Arcus PQ in eadem Ratione, et Vires V et v vel maneant, vel muten-

tur etiam in data aliqua Ratione. Detur postremo situs tum Orbitalium ad invicem, tum Corporum S, T, P, in his Orbitis, atque ob datum et situ et specie Triangulum S P T, et datam Rationem Virium, Lineola Qx semper erit similiter sita quoad hoc Triangulum: Erunt etiam in hoc casu Vires L M, M N ut Vis v , et Lineola Qx ut Lineola simul genita Q R, et Errores omnes ex Lineola Qx oriundi, ut Qx vel Q R, vel ut Diametri Orbitalium; atque adeo reddentur proportionales. Denique ob datam Rationem inter Arcum P Q et totam Orbitae P A B perimetrum, Tempus quo generatur Qx erit ut Tempus periodicum Corporis P circum T, vel Corporis T circum S. Sunt itaque Errores similes lineares ut Orbitalium Diametri, et propterea Errores angulares ex Centro T spectati aequales; et Errorum linearium similium vel Angularum aequalium Tempora sunt ut Tempora periodica.

In Corol. 16.

Sit T Tempus periodicum Telluris nostrae circa Solem, t Tempus periodicum Lunae circa Centrum Telluris, R Tempus periodicum Jovis circa Solem, a Tempus periodicum Satellitis alicujus circum-jovialis circa Centrum Jovis, et propositum sit Errores omnes Satellitis ex analogis Erroribus lunaribus Ope *Corol.* 14. et 15. derivare.

Manentibus Orbita et Tempore periodico Satellitis, deturbetur *Jupiter* de loco suo, et statuatur ad Distantiam a Sole, quae sit ad Distantiam Satellitis

litis a Centro Jovis ut Distantia Terrae nostrae a Sole ad Distantiam Lunae a Centro Terrae.

Sit S Tempus periodicum Jovis circa Solem ad hanc Distantiam revolvantis, eritque per *Corol.* 15.

r ad S ut t ad T ; unde $\frac{r}{S}$ aequalis erit $\frac{t}{T}$, et

$\frac{rr}{SS} = \frac{tt}{TT}$. Jam vero ex *Corol.* 15. manifestum

est Errores omnes in Motu Satellitis Tempore r commissos, aequales esse Erroribus similibus in motu Lunae commissis Tempore t . Referat itaque

que $\frac{t^2}{T^2}$ Quantitates Errorum periodicorum lunarium, et huic aequalis $\frac{rr}{SS}$ exhibebit Errores ana-

logos aequales in motu Satellitis. Restituatur jam Jupiter in locum proprium, et per *Corol.* 14. mutabuntur Quantitates Errorum in reciproca duplicata Ratione Temporis periodici Jovis circa

Solem, hoc est, in Ratione $\frac{I}{SS}$ ad $\frac{I}{RR}$: Quare

$\frac{rr}{RR}$ jam exponet Quantitates Errorum Satellitis, hoc est, Errores periodici Satellitis erunt ad Errores

analogos Lunares, ut $\frac{rr}{RR}$ ad $\frac{tt}{TT}$; omnino ut in *Corol.* 16. 16. alia Ratione demonstratum est.

Vires LM, MN, caeteris stantibus, sunt ut PT.]

Nam Vires LM, MN sunt ut $\frac{SK \times PT}{ST}$. Sed

si detur ST, dabitur SK ob datam Vim absolutam Corporis

Corporis S, quo in casu, dabitur $\frac{SK}{ST}$, et Vires LM, MN erunt ut PT.

Et motus uterque erit ut Tempus periodicum Corporis P directe, et Quadratum Temporis periodici Corporis T inverse.

Motus periodici sunt qui Spatio unius Revolutionis peraguntur, adeoque sunt ut motus contemporanei et Tempora periodica conjunctim; unde fit ut motus contemporanei sint ut motus periodici directe et Tempora periodica inverse.

Sed motus periodici sunt ut $\frac{t}{T}$; ergo motus contemporanei erunt ut $\frac{t}{T}$.

In Corol. 17.

Sit V Vis attractiva Corporis S, v Vis attractiva Corporis T; sit T Tempus periodicum Corporis T circum S, t Tempus periodicum Corporis P circum T, et Vis mediocris LM erit ad V ut PT ad ST: Est autem V ad v ut $\frac{ST}{TT}$ ad $\frac{TP}{tt}$; quare Vis mediocris LM est ad Vim v in Ratione composita ex Ratione PT ad ST, et ex Ratione $\frac{ST}{T^2}$ ad $\frac{TP}{tt}$. Sed hae duae Rationes componunt

Rationem $\frac{TP \times ST}{TT}$ ad $\frac{TP \times ST}{tt}$, hoc est, Ra-

tionem $\frac{1}{T}$ ad $\frac{1}{t}$ vel tt ad TT; quare Vis mediocris LM est ad Vim qua Corpus P retinetur

X

in

in Orbe suo ut Quadratum Temporis periodici Corporis P circum T ad Quadratum Temporis periodici Corporis T circum S.

Ad Prop. 4. Lib. 3.

Periodus Lunaris respectu fixarum est 27 dierum, 7 horarum, et 43 minutorum primorum; seu minutorum primorum 39343: Sed Vis qua Luna in Orbe suo retinetur minor est quam si a Terra solum manaret in Ratione $177\frac{2}{3}$ ad $178\frac{2}{3}$, seu 7109 ad 7149; adeoque si periodus Lunaris desideretur, qualis ab Attractione sola Telluris prodiret, minuenda est periodus 39343 minutorum primorum in dicta Ratione subduplicata, nempe in Ratione 7139 ad 7119, et prodibit minutorum primorum 39233. Sic itaque procedit calculus.

Ambitus Terrae 123249600 pedes Parisienses, Peripheria Orbitae Lunarise 7394976000, Arcus quem Luna medio Motu (ut supra correcto) Tempore unius minuti primi describit 188489, hujus Quadratum 35528103121, Diameter Telluris 39231600, Diameter Orbitae Lunarise 2353896000, Quadratum Arcus Tempore unius Minuti primi descripti ad hanc Diametrum applicatum $15,09\frac{1}{3}$; quae Longitudo est pedum Parisiensium 15,1 Digiti, et $1\frac{2}{3}$ Lineae.

Idem aliter.

Periodus Lunarise absque Diminutione 39343 minutorum primorum, Orbita lunarise 7394976000,

Arcus Tempore minuti unius primi descriptus 187962, Quadratum Arcus 35329713444, Diameter Orbitae Lunarise 235389600, Spatium rectilineum per quod descenditur Tempore Minuti unius primi apud Lunam, priusquam corrigatur 15,009037 pedum, Spatium correctum $15,09\frac{1}{3}$.

Vis qua Luna retinetur in Orbe suo, oritur potissimum ab Actione Terrae, sed nonnihil etiam ab Actione Solis; estque haec Vis composita minor quam si a Terra sola manaret in Ratione $177\frac{2}{3}$ ad $178\frac{2}{3}$; quare si desideretur Gravitas Lunae ad solam Terram, augendum est Spatium 15,009037 in Ratione $177\frac{2}{3}$ ad $178\frac{2}{3}$; atque haec est Correctio supra memorata. Vide *Corol. Prop. 3. Lib. 3.*

Actiones Solis et Lunae in Aestibus marinis excitandis esse, caeteris paribus, ut Vires eorum absolutae directe, et ut Cubus Distantiae eorum a Centro Terrae inverse.

Fig. 74.

Designet S Centrum Solis vel Lunae, T Centrum Telluris, ZN et HO Diametros terrestres se mutuo decussantes ad Angulos rectos in Centro T; ita ut locus Z habeat Punctum S pro Zenith: Designet etiam V Vim absolutam attractivam Corporis S, et Attractiones in locis Z, T et N erunt

ut $\frac{V}{SZ^2}$, $\frac{V}{ST^2}$, et $\frac{V}{SN^2}$ respective: et excessus

Attractionis in Z supra Attractionem in T, et

X 2

At-

Attractionis in T supra Attractionem in N, erunt

$\frac{V}{SZ^2} - \frac{V}{ST^2}$, et $\frac{V}{ST^2} - \frac{V}{SN^2}$ respective; ergo Vis qua elewantur aquae in loco Z erit $\frac{V}{SZ^2} - \frac{V}{ST^2}$, vel $\frac{V}{ST^2} - \frac{V}{SN^2}$, prout constitutum fuerit Corpus

S in Zenith vel Nadir hujus loci: Cape harum Virium semi-summam, et habebis Vim medio-crem qua excitantur aquae in loco Z, absque ulla habita Ratione situs Corporis S, sive in Zenith

versetur, sive in Nadir, nimirum $\frac{V}{2} \times \frac{1}{SZ^2} - \frac{1}{SN^2}$,

vel $\frac{V \times SN^2 - SZ^2}{2SN^2 \times SZ^2}$, vel $\frac{V \times SN + SZ \times SN - SZ}{2SN^2 \times SZ^2}$,

vel $\frac{V \times 2ST \times ZN}{2SN^2 \times SZ^2}$, vel $\frac{V \times ST \times ZN}{SN^2 \times SZ^2}$, vel

$\frac{V \times ST \times ZN}{ST^4}$, vel $\frac{V \times ZN}{ST^3}$; nam cum Longi-

tudo ST permagna sit prae Longitudine Semi-diametri TZ erit Rectangulum SN $\times$ SZ ad Quadratum Distantiae ST in Ratione Aequalitatis quam proxime.

Porro, si Vis attractiva Corporis S in loco H exponeretur per Longitudinem SH, Vis qua deprimuntur Aquae in loco H exponenda esset per Longitudinem HT; nam Vis SH idem valet atque Vires HT, TS; quarum Vis TS, trahendo particulas Aquarum secundum Rectas ipsi TS parallelas, nihil confert ad depressionem Aquarum, dum altera Vis HT tota impenditur in hoc negotium:

gotium : Sed Vis attractiva Corporis S non exponenda est per Longitudinem S H, sed per Quan-

titatem $\frac{V}{S H^2}$, vel $\frac{V \times S H}{S H^3}$, vel $\frac{V \times S H}{S T^3}$; (nam

cum Longitudo T H quam minima sit quoad Longitudinem S T, et ponatur Angulus S T H rectus, Ratio S H ad S T perexigua erit si conferatur cum Ratione S T ad S Z, atque adeo pro Ratione Aequalitatis tuto haberi potest); ergo depressio Aquarum in loco H, non amplius exponenda est per Longitudinem H T, sed per

Quantitatem $\frac{V \times H T}{S T^3}$; unde et ex computo priore

facile perspicitur, Elevationem Aquarum in loco Z duplo majorem esse, quam est depressio earum in loco H; et Actionem utramque (*caeteris paribus*), sicut et hinc pendentes Aestus marinos, esse ut Vis absoluta attractiva Corporis S directe, et Cubus Distantiae S T inverse. Q. E. D.

F I N I S.



ERRATA.

Pag. 56. for Prob. X. $\mathcal{C}$. read Prob. XI. $\mathcal{C}$.

86. l. 8. for B read A.

109. l. 10. for $\frac{FO}{CF}$ read $\frac{FO}{CF^3}$.

113. l. 11. for 2 P C read P C.

136. l. penult. for $z-\lambda^n$ read $z\lambda^n$.

148. l. 17. for $+ghz^n$ read $g+bz^n$.

Ibid. l. 19. for $fg+eb$ read $fg-eb$.

152. l. 7. for $-gkk$ read $+gkk$.

160. l. 5. for $-z^2$ read $-x^2$.

198. for vii. read viii.

If any other Mistakes have happened, we hope they are such as an intelligent and candid Reader will easily correct and forgive.

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